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A THEORETICAL INVESTIGATION OF ELECTROMAGNETIC WAVES OBLIQUELY INCIDENT UPON A PLASMA SLAB

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16. Abstract

The problem of an electromagnetic wave obliquely incident upon a plasma slab is considered as a boundary-value problem by use of a self-consistent solution of the coupled linearized Vlasov and Maxwell equations. Power reflection, transmission, and absorption coefficients are derived under the assumption that all particles undergo specular reflection at the surfaces of the plasma slab. Although the analysis is valid for arbitrary slab thickness, computational results are presented for slabs which are thin when compared with a wavelength. The results show that a series of resonances occur which are attributed to the finite temperature of the plasma. The results further show that the resonances are Landau damped as the thermal velocity of the plasma electrons increases. It is shown that similar resonances can be predicted from the coupled linearized hydrodynamic Maxwell equations; however, as is well known, such a model does not predict Landau damping. The effects of a finite collision frequency are then included by means of a simple Bhatnagar-Gross-Krook (BGK) collision term. The numerical computations vividly indicate that the resonances undergo severe damping for extremely small ratios of the collision frequency to the signal frequency.

Finally, the plasma capacitor problem is considered, and the results indicate that the longitudinal resonances have characteristics very similar to those of the plane-wave resonances.

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SUMMARY

The problem of an electromagnetic wave obliquely incident upon a plasma slab is considered as a boundary-value problem by use of a self-consistent solution of the coupled linearized Vlasov and Maxwell equations. Power reflection, transmission, and absorption coefficients are derived under the assumption that all particles undergo specular reflection at the surfaces of the plasma slab. Although the analysis is valid for arbitrary slab thickness, computational results are presented for slabs which are thin when compared with a wavelength. The results show that a series of resonances occur which are attributed to the finite temperature of the plasma. The results further show that the resonances are Landau damped as the thermal velocity of the plasma electrons increases. It is shown that similar resonances can be predicted from the coupled linearized hydrodynamic Maxwell equations; however, as is well known, such a model does not predict Landau damping. The effects of a finite collision frequency are then included by means of a simple Bhatnagar-Gross-Krook (BGK) collision term. The numerical computations vividly indicate that the resonances undergo severe damping for extremely small ratios of the collision frequency to the signal frequency.

Finally, the plasma capacitor problem is considered, and the results indicate that the longitudinal resonances have characteristics very similar to those of the plane-wave resonances.

INTRODUCTION

The interaction of electromagnetic waves with plasmas has been of continuing interest to those engaged in the study of the ionosphere, of the radar return from meteor trails, and of reentry plasma sheaths. In the last category, most of the research emphasis has been directed toward the solution of boundary-value problems that vary in complexity

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from relatively simple plane wave interactions (ref. 1) to rather complicated ones involving antennas under "cold" plasmas (ref. 2) and compressible plasmas (ref. 3).

A cold plasma is defined here as one in which the electron thermal velocity is zero, and thus the plasma behaves as an incompressible fluid which exerts no pressure. One of the primary shortcomings of this model is that no mechanism is provided in which to excite a spectrum of longitudinal plasma waves. The importance of these longitudinal oscillations lies in the fact that they have been observed experimentally in the laboratory, as far back as 1931 (ref. 4) and later in connection with the radar scattering from cylindrical plasma columns (ref. 5). It was observed that the radar return consisted of a series of resonances, the characteristics of which are shown in figure 1. The interesting feature of these resonances is that cold plasma theory predicts only the main resonance which, for the cylindrical column, occurs at $\omega_p = \sqrt{2}\omega$ where ω is the frequency of propagation and ω_p is the plasma frequency. (For a plane slab, this resonance occurs when $\omega = \omega_p$.) Much more comprehensive and convincing experiments were initiated in 1957 by Dattner (ref. 6). His experiment consisted of placing a small cylindrical discharge tube into the wall of a waveguide with the electric field perpendicular to the tube and monitoring the reflection coefficient as a function of increasing discharge tube current. Because of the thoroughness of the experiment, there is no doubt that the secondary resonances exist, and they have since that time been termed Tonks-Dattner resonances. Although excellent experimental results were available, a satisfactory analytical explanation for these resonances did not appear until the classic work of Parker, Nickel, and Gould (ref. 7) was published. Upon applying a fluid model of the plasma, they were able to conclude that a spectrum of resonances were generated at the frequencies

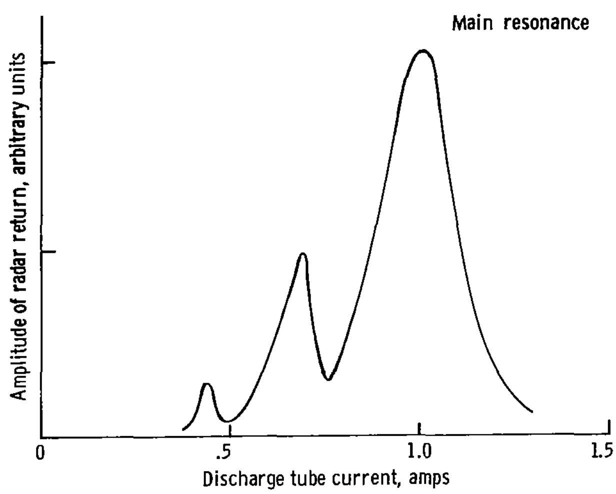


Figure 1.- Radar return from a cylindrical plasma column.

$$\omega^2 = \omega_p^2 + \frac{3KT_e}{m}k^2$$

where T_e is the electron temperature, K is the Boltzmann constant, m is the electron mass, and k is a wave number which depends on the radius of the plasma column. These frequencies correspond to longitudinal plasma oscillations, which couple more strongly to the electromagnetic wave because the phase velocity of such waves is of the order of the thermal velocity of the electrons in the plasma.

One of the shortcomings of an approach based on the linearized fluid equations lies in the fact that for finite thermal velocity, the fluid equations are valid only for $\omega \approx \omega_p$ (ref. 7). Analytical results based on such a model should therefore become less inaccurate as the ratio of ω_p/ω decreases. Another shortcoming is the failure of the fluid equations to predict Landau damping (ref. 8). The proper description requires a detailed solution of the Vlasov equation, where the Vlasov equation is identified as the collisionless Boltzmann equation to be solved self-consistently with Maxwell's equations. The importance of the Landau damping lies in the fact that the collisionless damping should be pronounced as the ratio of the thermal velocity to the phase velocity of the longitudinal wave increases. It will be shown that this ratio increases as the order of the resonance increases, and could account for the damping of the secondary resonances shown in figure 1. In order to determine how the widths of the resonances at half-maximum behave in detail as a function of all the parameters, it is necessary to solve a boundary-value problem. The model considered herein consists of a plane wave obliquely incident upon a plasma slab with the electric vector polarized in the plane of incidence so that longitudinal plasma oscillations are excited. This particular problem also has applications to the study of antennas under reentry plasmas, because the radiation characteristics of such antennas can be described by a spectrum of plane waves. (See ref. 9.) The kinetic treatment of this problem has previously been considered by Hinton (ref. 10) and by Bowman and Weston (ref. 11) in the United States and by Kondratenko and Miroshnichenko (ref. 12) in the Soviet Union. Hinton (ref. 10) solved the problem by expressing the currents as integrals over particle orbits. This procedure is equivalent to solving the Vlasov equation. The approach, however, requires several ponderous perturbation expansions and leads to an integral equation solution of the problem. Bowman and Weston (ref. 11), on the other hand, used the singular eigenfunction techniques of Case (ref. 13), Shure (ref. 14), Felderhof (ref. 15), and Van Kampen (ref. 16) to obtain solutions to the Vlasov-Maxwell equations. The disadvantage of this approach is that analytical and numerical results appear to be rather difficult to obtain. Kondratenko and Miroshnichenko (ref. 12) published an excellent and concise piece of work. Proceeding as Landau (ref. 8) did for the half-space problem, they used an integrating factor to solve the Vlasov equation. This procedure resulted in a solution in the form of an integral equation which was reduced by means of a Fourier series. The treatment presented herein differs from theirs, largely in the initial formulation procedure, although the mathematical results are the same.

In none of these papers were numerical results presented. In fact, the only computations which have appeared were done by Melnyk (ref. 17), who considered a plasma the equilibrium statistics of which are governed by degenerate Fermi-Dirac statistics. Maxwell-Boltzmann statistics will be considered and the problem will be approached by initially assuming specular reflection of electrons at the plasma boundaries. This procedure, as will be shown, automatically allows an immediate choice of a Fourier series representation of the problem. This procedure does not lead to a solution expressed in the form of an integral equation. It is in this way that the formulation herein differs from that in reference 12. The usual electromagnetic boundary conditions are used in connection with the boundary condition of specular reflection. The reflection, transmission, and absorption coefficients are then solved for and calculated as functions of the plasma electron density and thermal velocity for a slab which is thin compared with a free-space wavelength and for zero collision frequency. A series of resonances, that is, peaks in the reflection coefficient, occur which exhibit features of the Tonks-Dattner resonances, and which become Landau damped as the thermal velocity of the plasma increases. The reflection coefficient described by a continuous fluid model of a plasma is also computed; similar resonances are noted except that they are not Landau damped.

A kinetic analysis of the plasma capacitor (ref. 18) is included to strengthen the physical deduction concerning the predominance of longitudinal oscillations in the plane-wave solution. The results show that the plasma capacitor, which contains only a longitudinal electric field, resonates at precisely the same slab thickness, plasma frequency, thermal velocity, and propagating frequency as those for the plane wave interacting with the slab. These resonances are more conventionally defined in the sense that a peak in resistance and a zero in reactance are noted at the resonant frequency.

Finally, a finite collision frequency is considered by using a simple Bhatnagar-Gross-Krook (BGK) collision term (ref. 19), and for purposes of nomenclature, the kinetic equation will be referred to as the Vlasov equation. The results show that the higher order resonances are completely damped out at such a small value of the ratio of collision to propagating frequency that laboratory reproduction of such resonances would be difficult to achieve at normal radio and microwave frequencies. It is concluded that although the present model exhibits some characteristics of the Tonks-Dattner resonances on a qualitative basis, the detailed structure of the resonances is influenced by another mechanism, probably the inhomogeneity of the plasma.

SYMBOLS

A	power absorption coefficient
a	speed of sound in electron gas, $\sqrt{\gamma}v_T$

B magnetic flux density

$b = (\omega - k_o v_z \sin \theta)/v_x$

C capacitance

c speed of light

D dielectric displacement

E electric field intensity

E_o amplitude of incident electric electric field intensity

e electronic charge

F_o normalized probability function (assumed to be Gaussian in this paper)

$F^+ = f^+ + f^-$

$F^- = f^+ - f^-$

f^+ distribution function for particles having velocity v_x greater than zero

f^- distribution function for particles having velocity v_x less than zero

f distribution function (no velocity half-plane restrictions)

G_1, G_2 impedance coefficients for parallel polarization defined in equation (15)

$G_1^{\perp}, G_2^{\perp}$ impedance coefficients for perpendicular polarization defined in equation (79)

H magnetic field intensity

H_o amplitude of incident magnetic field intensity

H_1, H_2^* Fourier transforms of two variables in velocity space, defined in equations (62)

I/A current per unit area

$i = \sqrt{-1}$

$J_l, J_{1l}, J_{2l}, J_{3l}, J_{4l}, J_{5l}$ dispersion integrals defined by equation (A1)

j current density

K Boltzmann constant

k wave number

k_o wave number in free space

k_p x-component of wave number in a cold plasma; $k_p = k_o\sqrt{\frac{\epsilon}{\epsilon_o} - \sin^2\theta}$ for $\omega_p/\omega < \cos\theta$ and $k_p = ik_o\sqrt{\sin^2\theta - \frac{\epsilon}{\epsilon_o}}$ for $\omega_p/\omega > \cos\theta$

k_u wave number of longitudinal plasma oscillations; $k_u = k_o\sqrt{\frac{1}{a^2/c^2}\frac{\epsilon}{\epsilon_o} - \sin^2\theta}$ for $\frac{\sqrt{\epsilon/\epsilon_o}}{a/c} > \sin\theta$ and $k_u = ik_o\sqrt{\sin^2\theta - \frac{\epsilon/\epsilon_o}{a^2/c^2}}$ for $\frac{\sqrt{\epsilon/\epsilon_o}}{a/c} < \sin\theta$

L thickness of plasma slab

l Fourier indices ($l = 0,1,2,3,\ldots$)

m mass of electron

n electron number density

P pressure

R reflection coefficient or resistance, as appropriate

T transmission coefficient

T_e electron temperature

t	time
u	fluid velocity
$\vec{u}_x, \vec{u}_y, \vec{u}_z$	unit vectors in three principal directions
V	voltage
v	particle velocity
v_{ph}	phase velocity, ω/k
v_T	root-mean-square particle velocity, $\sqrt{\frac{KT_e}{m}}$
X	reactance
X_o	reactance of air-filled parallel-plate capacitor, $1/\omega C$
x,y,z	Cartesian coordinates
x'	dummy variable for integration
Z	impedance
γ	constant in adiabatic equation, $Pn^{-\gamma} = \text{Constant}$
δ_i^j	Kronecker delta, 0 when $i \neq j$ and 1 when $i = j$
ϵ	permittivity
ϵ_o	permittivity of free space
ζ	argument of dispersion function, $\frac{\omega L}{l\pi} \frac{1}{\sqrt{2}v_T} \frac{1}{\sqrt{1 + \frac{k_o^2 L^2 \sin^2\theta}{l^2\pi^2}}}$
ζ_o	value of ζ for $l = 0$
ζ'	value of ζ for case of a nonzero collision frequency
θ	angle of incidence

κ	wave number, $l\pi/L$
Λ	velocity transform variable
$\Lambda^+ = 1 - \frac{\omega_p^2}{v_T^2}\frac{L}{l\pi}J_{1l}$	
μ_o	permeability of free space
ν	collision frequency
σ	conductivity
ω	angular frequency of propagation
ω_p	angular plasma frequency
$\rightarrow$	vector
$\longleftrightarrow$	tensor
$\vert\ \vert$	absolute value
∇	del operator
[ ]	matrix
$\overline{}$	Fourier transform

Subscripts:

x,y,z	denotes vector components along the principal directions
l	denotes Fourier components
o	denotes first-order or unperturbed quantity, unless otherwise specified
1	denotes second-order or perturbed quantity, unless otherwise specified

$\perp$ denotes component of reflection or transmission coefficient perpendicular to the plane of incidence

$||$ denotes component of reflection or transmission coefficient parallel to the plane of incidence

INTERACTION OF A PLANE WAVE WITH A UNIFORM PLASMA SLAB

Figure 2 shows the geometry of the problem. A plane wave is incident upon a plasma slab with the electric vector polarized in the plane of incidence. The incident electromagnetic wave is assumed to have a harmonic dependence of the form

$$E = E_o \exp i\left(k_o x \cos \theta + k_o z \sin \theta - \omega t\right)$$

The faces of the plasma slab are $x = 0$ and $x = L$; the plane of incidence is the x,z plane; and the angle of incidence is θ. Here k_o is the free-space propagation constant $k_o = \omega/c$ and ω is the angular frequency of the incident wave. The case where the electric vector of the incident wave is perpendicular to the plane of incidence is discussed in an analogous manner in a later section. Kinetic effects, however, depend upon the ratio of thermal velocity to the phase velocity of the plasma waves involved. This ratio is appreciable only for longitudinal plasma waves. These waves, however, do not couple to incident electromagnetic waves polarized perpendicular to the plane of incidence. The reflection coefficient for the case where the electric vector is parallel to the plane of incidence is discussed, first for the linearized cold plasma model, then for the linearized fluid model, and finally for the linearized Vlasov equation (with a BGK collision term). The equations describing the plasma in each case are solved self-consistently with Maxwell's equations. In each case, the tangential field components at the left of the slab $(x < 0)$ are given by

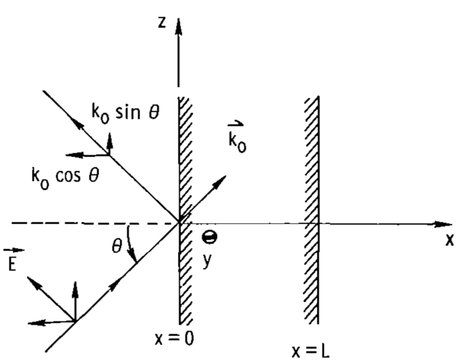


Figure 2.- Geometry of a plane wave obliquely incident upon a plasma slab.

$$H_y = H_o\left(e^{ik_o x \cos \theta} + Re^{-ik_o x \cos \theta}\right)e^{i\left(k_o z \sin \theta - \omega t\right)} \tag{1}$$

$$E_z = -H_o\left(e^{ik_ox \cos\theta} - Re^{-ik_ox \cos\theta}\right)\sqrt{\frac{\mu_o}{\epsilon_o}} \cos\theta \, e^{i\left(k_oz \sin\theta - \omega t\right)} \tag{2}$$

where R is the complex reflection coefficient for the magnetic field and H_o is the magnetic field amplitude of the incident wave. MKSA units are used throughout the paper.

To the right of the slab ($x > L$),

$$H_y = H_oTe^{ik_ox \cos\theta}e^{i\left(k_oz \sin\theta - \omega t\right)} \tag{3}$$

and

$$E_z = -H_oT\sqrt{\frac{\mu_o}{\epsilon_o}}e^{ik_ox \cos\theta}e^{i\left(k_oz \sin\theta - \omega t\right)}\cos\theta \tag{4}$$

Here T is the complex transmission coefficient for the magnetic field. The boundary conditions across the surfaces $x = 0$ and $x = L$ require that the z-dependence of the fields within the slab be the same as those outside; therefore, the fields inside the slab are of the form $[E,H] = [E(x),H(x)]\,e^{i\left(k_oz \sin\theta - \omega t\right)}$. As such, the exponential dependence $e^{i\left(k_oz \sin\theta - \omega t\right)}$ need not explicitly appear in any of the subsequent expressions.

Interaction of a Plane Wave With a Uniform Cold Plasma Slab

If the plasma is cold, the random velocity of the free electrons is assumed to be zero, and the dielectric constant of the plasma can be determined without resorting to kinetic theory. The equations of motion of a free electron interacting with an electromagnetic wave are solved in order to deduce the polarization per particle; from the polarization the following expression for the relative dielectric constant of the plasma is obtained:

$$\frac{\epsilon}{\epsilon_o} = 1 - \frac{\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} + i\frac{\left(\frac{\nu}{\omega}\right)\left(\frac{\omega_p}{\omega}\right)^2}{1 + \left(\frac{\nu}{\omega}\right)^2} \tag{5}$$

where ω_p is the plasma frequency, $\omega_p{}^2 = \frac{n_oe^2}{m\epsilon_o}$ (n_o is the electron density, m is the electron mass, e is the electron charge, and ϵ_o is the permittivity of free space) and ν is the collision frequency for momentum transfer. The plasma frequency ω_p and the collision frequency ν are assumed to be constant.

The solution to this problem appears in reference 20 but not in a form which will be useful when these results are compared with those obtained for the fluid and kinetic models.

In order to develop the desired solution the Maxwell curl equations

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = -i\omega\epsilon \vec{E} \tag{6}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_o \vec{H} \tag{7}$$

are used. Since the plasma is nonparamagnetic, the permeability of the plasma μ is assumed to be that of free space μ_o, and the dielectric constant ϵ is given by equation (5). By using equations (6) and (7), $\vec{H} = H_y(x)\, \vec{u}_y$ in the slab $\left(\vec{u}_y\right.$ is the unit vector in the y-direction$\left.\right)$ can formally be derived. The result is

$$H_y(x) = \frac{H_y(L) \sin k_p x - H_y(0) \sin k_p(x - L)}{\sin k_p L} \tag{8}$$

where $H_y(0)$ and $H_y(L)$ are the values of H_y at $x = 0$ and $x = L$, respectively, and $k_p = k_o\sqrt{\frac{\epsilon}{\epsilon_o} - \sin^2\theta}$ for $\omega_p/\omega < \cos\theta$ and $k_p = ik_o\sqrt{\sin^2\theta - \frac{\epsilon}{\epsilon_o}}$ for $\omega_p/\omega > \cos\theta$. Reflection, transmission, and absorption coefficients can now be determined from the boundary conditions, that is, continuity of tangential E and tangential H at $x = 0$ and $x = L$. Use of equations (1) to (4), (6), (8), and the boundary conditions leads to the following relationships:

$$(1 + R)H_o = H_y(0) \tag{9}$$

$$H_o T e^{ik_o L \cos\theta} = H_y(L) \tag{10}$$

$$-\sqrt{\frac{\mu_o}{\epsilon_o}}(1 - R)H_o \cos\theta = -i\omega\mu_o\left[H_y(0)\, G_1 L - H_y(L)\, G_2 L\right] \tag{11}$$

$$-\sqrt{\frac{\mu_o}{\epsilon_o}} H_o T e^{ik_o L \cos\theta} \cos\theta = -i\omega\mu_o\left[H_y(0)\, G_2 L - H_y(L)\, G_1 L\right] \tag{12}$$

It is important to note here that the functions G_1 and G_2 are defined separately for (a) the cold plasma model, (b) the fluid model, and (c) the Vlasov model. In this way, a single algebraic relationship can be used to solve equations (9) to (12) once G_1 and G_2 are given. For the cold plasma,

$$G_1 = \left[\left(\frac{k_p}{k_o}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_o}}\right] \frac{\cot k_pL}{k_pL} \tag{13}$$

$$G_2 = \frac{G_1}{\cos k_pL} = \left[\left(\frac{k_p}{k_o}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_o}}\right] \frac{\csc k_pL}{k_pL} \tag{14}$$

If Z is defined as the surface impedance of the plasma at $x = 0$,

$$Z = \frac{ik_oL}{\cos\theta} G_1 - \frac{G_2^2}{G_1 - \frac{i\cos\theta}{k_oL}} \tag{15}$$

It follows then that

$$R = \frac{1 - Z}{1 + Z} \tag{16}$$

and

$$T = \frac{(1 + R)G_2 e^{-ik_oL\cos\theta}}{G_1 - \frac{i\cos\theta}{k_oL}} \tag{17}$$

The absorption coefficient may be defined as:

$$A = 1 - |R|^2 - |T|^2 \tag{18}$$

In the absence of collisions ($\nu = 0$), the absorption coefficient of the cold plasma slab is zero.

In the limit as L approaches ∞, only forward-traveling waves exist in the plasma slab, and the expression for the surface impedance reduces to

$$Z_{L\to\infty} = \frac{k_p}{k_o} \frac{1}{\frac{\epsilon}{\epsilon_o}} \frac{1}{\cos\theta} \tag{19}$$

Near $(\omega_p/\omega)^2 = 1$ and $(\nu/\omega) \approx 0$, the dielectric constant approaches zero, and G_1 and G_2 become large; thus, the surface impedance can approximately be written as

$$Z_{\epsilon\to 0} = -\left(\frac{k_p}{k_o}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_o}} \frac{\tan k_pL}{k_pL} \frac{ik_oL}{\cos\theta} \tag{20}$$

which for a thin slab $\left(k_pL\right.$ approaching $\left.0\right)$ reduces to

$$Z_{\substack{\epsilon\to 0\\ L\to 0}} = -\frac{ik_oL}{\cos\theta}\left(\frac{k_p}{k_o}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_o}} \tag{21}$$

It is seen that the reflection coefficient of even a thin slab, for which $\nu = 0$ and ϵ approaches 0, should be unity at $\omega = \omega_p$ because the impedance becomes infinite. Also note that the limit given by equation (21) depends upon the order in which the limits are taken with respect to ϵ and L. Naturally, for $L \equiv 0$, Z goes to one, not zero as implied by equation (21). The thin slab is investigated in more detail later. When the slab is not thin, from equations (13), (14), and (15) it can be seen that the impedance approaches zero when $k_pL = \left(l + \frac{1}{2}\right)\pi$ for $l = 0,1,2,\ldots$. These values are the Fabry-Perot resonances which are familiar in optics. Numerical results are given in a subsequent section.

Fluid Description of Plane Wave Problem

The linearized Vlasov equation, with a BGK collision term of the form $-\nu\left(f - f_o\right) = -\nu f_1$ (where f_o is the unperturbed distribution function and f_1 is the perturbation) and with $\partial/\partial t = -i\omega$, may be written as

$$-i\omega f_1 + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} - \frac{e\vec{E}}{m} \cdot \frac{\partial f_o}{\partial \vec{v}} = -\nu f_1 \tag{22}$$

If the zeroeth and first moments of equation (22) are calculated with respect to the particle velocity, the following expressions are obtained (see ref. 21, for example) for conservation of mass and momentum:

$$-i\omega n_1 + n_o\nabla \cdot \vec{u} = 0 \tag{23}$$

$$(-i\omega + \nu)n_o\vec{u} = -\frac{n_oe\vec{E}}{m} - \nabla \cdot \overleftrightarrow{P} \tag{24}$$

where $\vec{u}$ is the fluid velocity, $\overleftrightarrow{P}$ is the pressure tensor, and n_o and n_1 are the unperturbed electron density and its perturbation, respectively.

The pressure tensor corresponds to terms in the next higher moment, which can be eliminated by assuming a scalar pressure and using the equation of state

$$P = nKT_e \tag{25}$$

(K is the Boltzmann constant and T_e is the electron temperature) in connection with an adiabatic equation

$$\frac{P}{n^\gamma} = \text{Constant} \tag{26}$$

If the pressure term in equation (24) is linearized, and n_1 in equation (23) is eliminated by using $P_1 = \gamma n_1 KT_e$, the following equations are obtained:

$$mn_o(\nu - i\omega)\vec{u} = n_o e\vec{E} - \nabla P_1 \tag{27}$$

$$a^2 mn_o \nabla \cdot \vec{u} = i\omega P_1 \tag{28}$$

where $a^2 = \gamma v_T{}^2 = \gamma KT_e/m$ and v_T is the thermal velocity. Equations (27) and (28) are the same ones Wait (ref. 3) used to derive the reflection coefficients for a uniform half-space. The procedure herein is similar to his, except that an additional boundary at $x = L$ is included. The electric field and the fluid velocity are related by the Maxwell equations

$$\nabla \times \vec{E} = i\omega\mu_o\vec{H} \tag{29}$$

$$\nabla \times \vec{H} = -i\omega\epsilon_o\vec{E} + n_o e\vec{u} \tag{30}$$

when the last term in equation (30) is the macroscopic convection current.

Equations (27) to (30) can be used to develop wave equations for P_1 and $\vec{H}$. Since the wave equation is a second-order differential equation, a total of four unknown coefficients must be determined within the slab (two coefficients for P_1 and two for $\vec{H}$). However, equations (27) and (29) can be used to show that the boundary condition of specular reflection, that is, $u = 0$ at $x = 0$ and $x = L$, implies that

$$\left.\frac{\partial H_y}{\partial z}\right|_{\substack{x=0\\x=L}} = \frac{i\omega\epsilon_o}{n_o e}\left.\frac{\partial P_1}{\partial x}\right|_{\substack{x=0\\x=L}}$$

The two unknown coefficients for P_1 can therefore be expressed in terms of those for $\vec{H}$ in the slab. This procedure leads to the following solutions:

$$H_y(x) = \frac{H_y(L) \sin k_p x - H_y(0) \sin k_p(x - L)}{\sin k_p L} \tag{31}$$

$$P_1(x) = \frac{\omega_p^2}{\omega^2} \frac{\omega k_o \sin \theta}{\frac{e}{mk_u}} \frac{\left[H_y(0) \cos k_u(x - L) - H_y(L) \cos k_u x\right]}{\sin k_u L} \tag{32}$$

where

$$k_u^2 = \frac{\omega^2}{a^2}\left(1 - \frac{\omega_p^2}{\omega^2} + i\frac{\nu}{\omega}\right) - k_o^2 \sin^2\theta \tag{33}$$

Note that the expression for $H_y(x)$ is identical to that obtained for the cold plasma. It also follows that

$$E_z = \frac{i\omega\mu_o}{k_o^2}\left\{H_y(0)\left[-\frac{k_p \cos k_p(x - L)}{\frac{\epsilon}{\epsilon_o} \sin k_p L} - \frac{\frac{\omega_p^2}{\omega^2} k_o^2 \sin^2\theta \cos k_u(x - L)}{\left(1 + i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_o} k_u \sin k_u L}\right]\right.$$

$$\left. + H_y(L)\left[\frac{k_p \cos k_p x}{\frac{\epsilon}{\epsilon_o} \sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2} k_o^2 \sin^2\theta \cos k_u x}{\left(1 + i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_o} k_u \sin k_u L}\right]\right\} \tag{34}$$

Therefore, if

$$G_1 = \left(\frac{k_p}{k_o}\right)^2 \frac{\cos k_p L}{\frac{\epsilon}{\epsilon_o} k_p L \sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2} \sin^2\theta \cos k_u L}{\left(1 + i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_o} k_u L \sin k_u L} \tag{35}$$

and

$$G_2 = \left(\frac{k_p}{k_o}\right)^2 \frac{1}{\frac{\epsilon}{\epsilon_o} k_p L \sin k_p L} + \frac{\frac{\omega_p^2}{\omega^2} \sin^2\theta}{\left(1 + i\frac{\nu}{\omega}\right)\frac{\epsilon}{\epsilon_o} k_u L \sin k_u L} \tag{36}$$

an expression for the impedance at $x = 0$ identical with the cold plasma result (eq. (15)) is obtained, but with the functions G_1 and G_2 given now by equations (35) and (36). When $k_u = \infty$, that is, when $a^2 = 0$, G_1 and G_2 reduce to those for the cold plasma. By inspecting equations (35) and (36), it is noted that for $\nu = 0$, Fabry-Perot type resonances occur when $k_u L = l\pi$ $(l = 0,1,2,\ldots)$ in addition to the cold plasma resonances $\epsilon/\epsilon_o = 0$ and $k_p L = (l + 1/2)\pi$ $(l = 0,1,2,\ldots)$. It is interesting to note at this point that the phase velocity can be very low for longitudinal waves, and therefore resonances can be expected for slabs which are thin compared with a free-space wavelength. If the thermal velocity $v_T = a/\sqrt{\gamma}$ is small compared with the speed of light, these resonances occur when

$$k_u L = n\pi \approx \frac{k_o L}{\sqrt{\gamma}\,\frac{v_T}{c}}\sqrt{1 - \frac{\omega_p{}^2}{\omega^2}} \tag{37}$$

or at a phase velocity $v_{ph} = \frac{k_o L}{n\pi} c$. The phase velocity of longitudinal waves therefore becomes smaller as the slab dimension decreases and as the order of the resonances increases. The quantity γ is normally assumed to be three for an electron gas.

Results of calculations for the reflection and absorption coefficients are presented in the section "Numerical Results," and comparisons are made with those obtained from the other models of the plasma.

Direct Solution of Plane Wave Problem
Using Linearized Vlasov Equation

In the preceding section, the linearized Vlasov equation (eq. (22)) was written with a BGK collision term of the form $-\nu(f - f_o) = -\nu f_1$ and with $\partial/\partial t = -i\omega$ as

$$-i\omega f_1 + \vec{v} \cdot \frac{\partial f_1}{\partial \vec{x}} - \frac{e\vec{E}}{m} \cdot \frac{\partial f_o}{\partial \vec{v}} = -\nu f_1$$

where f_o is the unperturbed distribution function, f_1 is the perturbation of the distribution function, $\vec{v}$ is the particle velocity, and $\vec{x}$ is the particle position. Equation (22) is to be solved self-consistently with the Maxwell curl equations:

$$\nabla \times \vec{H} = -i\omega\epsilon_o\vec{E} - e\int f_1\vec{v}\, d\vec{v} \tag{38}$$

$$\nabla \times \vec{E} = i\omega\mu_o\vec{H} \tag{39}$$

where the last term in equation (38) is the density of the convection current. In order to proceed, the distribution functions for the velocity half-space $v_x > 0$ and the velocity half-space $v_x < 0$ are considered separately. If the former is denoted by $f_1^+(\ldots v_x \ldots)$ and the latter denoted by $f_1^-(\ldots -v_x \ldots)$, f_1^+ and f_1^- satisfy the following equations:

$$(-i\omega + \nu)f_1^+ + v_x \frac{\partial f_1^+}{\partial x} + ik_o v_z \sin\theta\, f_1^+ - \frac{e}{m}\left(E_z \frac{\partial f_o}{\partial v_z} + E_x \frac{\partial f_o}{\partial v_x}\right) = 0 \tag{40}$$

$$(-i\omega + \nu)f_1^- - v_x \frac{\partial f_1^-}{\partial x} + ik_o v_z \sin\theta\, f_1^- - \frac{e}{m}\left(E_z \frac{\partial f_o}{\partial v_z} - E_x \frac{\partial f_o}{\partial v_x}\right) = 0 \tag{41}$$

If the expressions $F^+ = f_1^+ + f_1^-$ and $F^- = f_1^+ - f_1^-$ are introduced, the following second-order differential equation in x is obtained for F^-:

$$\frac{\partial^2 F^-}{\partial x^2} + \frac{(\omega + i\nu - k_o v_z \sin\theta)^2 F^-}{v_x^2} - \frac{2ie(\omega + i\nu - k_o v_z \sin\theta)}{m v_x^2}\frac{\partial f_o}{\partial v_x}E_x - \frac{2e}{m v_x}\frac{\partial f_o}{\partial v_z}\frac{\partial E_z}{\partial x} = 0 \tag{42}$$

If all particles are specularly reflected at $x = 0$ and $x = L$, then F^- must vanish at $x = 0$ and $x = L$. This condition can be satisfied identically by a Fourier sine series for F^- as a function of x:

$$F^- = \sum_{l=1}^{\infty} F_l(\vec{v}) \sin\frac{l\pi x}{L} \tag{43}$$

with

$$F_l^- = \frac{2}{L}\int_0^L F^- \sin\frac{l\pi x}{L}\, dx \tag{44}$$

Examining equations (42) and (43) indicates that they imply a Fourier sine series expansion for E_x and a cosine series for E_z.

If E_{lx}, E_{lz}, and H_{ly} are the corresponding Fourier coefficients for E_x, E_z, and H_y,

$$F_l^- = \frac{2ie\omega v_x F_o\left[\sqrt{\frac{\mu_o}{\epsilon_o}}\frac{v_z}{c}H_{ly} - \left(1 + i\frac{\nu}{\omega}\right)E_{lx}\right]}{m v_T^2\left[\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2\right]} \tag{45}$$

where

$$f_o = n_o F_o = \frac{n_o e^{-v^2/2v_T^2}}{\left(2\pi v_T^2\right)^{3/2}}$$

is the equilibrium distribution function and is assumed to be Maxwellian. The Fourier components of F^+ are derived in a similar manner. Since the x and z components of the current density are defined as

$$j_x = -e \iint_{-\infty}^{\infty} dv_z\, dv_y \int_0^{\infty} v_x F^-\, dv_x$$

and

$$j_z = -e \iint_{-\infty}^{\infty} dv_z\, dv_y\, v_z \int_0^{\infty} F^+\, dv_x$$

it follows that the Fourier components of j_x and j_z are:

$$\left.\begin{aligned} j_{lx} &= \frac{i\omega_p^2\omega\epsilon_o}{v_T^2}\frac{L}{l\pi}\left(J_{1l}E_{lx} - \sqrt{\frac{\mu_o}{\epsilon_o}}\frac{J_{2l}}{c}H_{ly}\right) \\ j_{lz} &= \frac{\omega_p^2\omega\epsilon_o}{v_T^2}\frac{L}{l\pi}\left(J_{3l}E_{lx} - \sqrt{\frac{\mu_o}{\epsilon_o}}\frac{J_{4l}}{c}H_{ly}\right) \end{aligned}\right\} \tag{46}$$

where

$$J_{1l} = \left(1 + i\frac{\nu}{\omega}\right)\frac{2l\pi}{L}\iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{v_x^2 F_o\, dv_x}{\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2} \tag{47}$$

$$J_{2l} = \frac{2l\pi}{L} \iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{v_x^2 v_z F_o\, dv_x}{\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2} \tag{48}$$

$$J_{3l} = 2\left(1 + i\frac{\nu}{\omega}\right) \iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{v_z\left(\omega - k_o v_z \sin\theta\right) F_o\, dv_x}{\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2} \tag{49}$$

$$J_{4l} = 2 \iint_{-\infty}^{\infty} dv_y\, dv_z \int_0^{\infty} \frac{dv_x\, v_z^2\left(\omega - k_o v_z \sin\theta\right) F_o}{\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2} \tag{50}$$

A Fourier analysis of the Maxwell curl equations (eqs. (38) and (39)) gives

$$\frac{l\pi}{L} E_{lz} = -ik_o \sin\theta\, E_{lx} + i\omega\mu_o H_{ly} \tag{51}$$

$$E_{lx} = \frac{k_o \sin\theta}{\omega\epsilon_o} H_{ly} + \frac{1}{i\omega\epsilon_o} j_{lx} \tag{52}$$

$$H_{ly}\left[k_o^2 - k_o^2 \sin^2\theta - \left(\frac{l\pi}{L}\right)^2\right] + \frac{l\pi}{L}\left[\frac{H_y(0) - (-1)^l H_y(L)}{L/2}\right] = -ik_o \sin\theta\, j_{lx} - \frac{l\pi}{L} j_{lz} \tag{53}$$

Therefore, if the modal components of the current (eq. (46)) are substituted into equations (52) and (53), three equations which contain the unknowns E_{lx}, E_{lz}, and H_{ly} are obtained. Solution of these equations yields:

$$H_{ly} = \frac{-\frac{l\pi}{L}\left[\frac{H_y(0) - (-1)^l H_y(L)}{L/2}\right]}{k_o^2 - k_o^2 \sin^2\theta - \left(\frac{l\pi}{L}\right)^2 - k_o^2 \frac{\omega_p^2}{\omega^2} \frac{\omega}{v_T^2}\left(J_{4l} - \frac{k_o L \sin\theta}{l\pi} J_{2l}\right)} \tag{54}$$

$$E_{lx} = \frac{\dfrac{l\pi}{\omega\epsilon_o L}\left[\dfrac{H_y(0) - (-1)^l H_y(L)}{L/2}\right]\left(k_o \sin\theta - \dfrac{\omega_p^2}{v_T^2}\dfrac{k_o L}{l\pi}\dfrac{J_{2l}}{c}\right)}{\left(1 - \dfrac{\omega_p^2}{v_T^2}\dfrac{L}{l\pi}J_{1l}\right)\left[k_o^2 - k_o^2\sin^2\theta - \left(\dfrac{l\pi}{L}\right)^2 - k_o^2\dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}\left(J_{4l} - \dfrac{k_o L}{l\pi}\sin\theta\, J_{2l}\right)\right]} \tag{55}$$

$$E_{lz} = \frac{i\omega\mu_o\left[\dfrac{H_y(0) - (-1)^l H_y(L)}{L/2}\right]\left(1 - \sin^2\theta - \dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}J_{5l}\right)}{\left(1 - \dfrac{\omega_p^2}{v_T^2}\dfrac{L}{l\pi}J_{1l}\right)\left[k_o^2 - k_o^2\sin^2\theta - \left(\dfrac{l\pi}{L}\right)^2 - k_o^2\dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}\left(J_{4l} - \dfrac{k_o L}{l\pi}\sin\theta\, J_{2l}\right)\right]} \tag{56}$$

By using equations (33) to (35),

$$E_z(0) = \sum_{l=0}^{\infty} E_{lz} = -i\omega\mu_o\left[H_y(0)\, G_1\, L - H_y(L)\, G_2 L\right] \tag{57}$$

where G_1 and G_2 are the functions previously introduced to define the surface impedance for the cold plasma and fluid models. From equations (54), (56), and (57), it follows that G_1 and G_2 for the Vlasov model are

$$G_1 = \sum_{l=0}^{\infty} \frac{\dfrac{2}{1+\delta_o^l}\left(\dfrac{1}{k_o L}\right)^2\left(1 - \sin^2\theta - \dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}J_{5l}\right)}{\left(1 - \dfrac{\omega_p^2}{v_T^2}\dfrac{L}{l\pi}J_{1l}\right)\left[1 - \sin^2\theta - \left(\dfrac{l\pi}{k_o L}\right)^2 - \dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}\left(J_{4l} - \dfrac{k_o L}{l\pi}\sin\theta\, J_{2l}\right)\right]} \tag{58}$$

$$G_2 = \sum_{l=0}^{\infty} \frac{\dfrac{2}{1+\delta_o^l}\left(\dfrac{1}{k_o L}\right)^2\left(1 - \sin^2\theta - \dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}J_{5l}\right)(-1)^l}{\left(1 - \dfrac{\omega_p^2}{v_T^2}\dfrac{L}{l\pi}J_{1l}\right)\left[1 - \sin^2\theta - \left(\dfrac{l\pi}{k_o L}\right)^2 - \dfrac{\omega_p^2}{\omega^2}\dfrac{\omega}{v_T^2}\left(J_{4l} - \dfrac{k_o L}{l\pi}\sin\theta\, J_{2l}\right)\right]} \tag{59}$$

where J_{5l} is defined as $J_{5l} = \left(J_{1l} - \dfrac{J_{2l}}{c}\sin\theta\right)\dfrac{\omega L}{l\pi}$. The reflection and transmission coefficients can be derived by substituting G_1 and G_2 into equations (15), (16), and (17).

Before proceeding, it is instructive to examine, in more detail, the dispersion integrals which appear in equations (58) and (59). Since it has been assumed that the unperturbed distribution function is Maxwellian, it follows that

$$\int_{-\infty}^{\infty} dv_y \; F_o\left(v_x,v_y,v_z\right) = F_o\left(v_x,v_z\right)$$

and the v_y integration trivially disappears. The integration on v_x may be extended over the entire real axis by noting that the integrand is an even function of v_x. Furthermore, if a partial fraction expansion is used, the integral J_{1l} becomes

$$J_{1l} = \left(1 + i\frac{\nu}{\omega}\right)\iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_o v_x}{\omega - k_o v_z \sin\theta - \dfrac{l\pi v_x}{L} + i\nu} \tag{60}$$

with similar reductions for the other dispersion integrals.

Use can now be made of Fourier transforms in velocity space (ref. 22) to reduce them to single integrals. For example, the zeroeth-order dispersion integral J_l can be written as a convolution in the form

$$J_l = \left(1 + i\frac{\nu}{\omega}\right)\iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_o}{\omega - k_o v_z \sin\theta - \dfrac{l\pi v_x}{L} + i\nu} \equiv \frac{\left(1 + i\frac{\nu}{\omega}\right)}{(2\pi)^2}\iint_{-\infty}^{\infty} H_1\left(\Lambda_x,\Lambda_z\right) H_2^*\left(\Lambda_x,\Lambda_z\right) d\Lambda_x \, d\Lambda_z \tag{61}$$

where Λ_x and Λ_z are the transform variables of the velocity components v_x and v_z, respectively, and $H_1\left(\Lambda_x,\Lambda_z\right)$ and $H_2^*\left(\Lambda_x,\Lambda_z\right)$ are given by

$$\left.\begin{aligned}
H_1\left(\Lambda_x,\Lambda_z\right) &= \frac{L}{l\pi}\iint_{-\infty}^{\infty} \frac{e^{i\left(\Lambda_x v_x + \Lambda_z v_z\right)}}{v_x - \left(\dfrac{L\omega}{l\pi} - v_z\dfrac{k_o L \sin\theta}{l\pi} + \dfrac{iL\nu}{l\pi}\right)} dv_x \, dv_z \\
H_2^*\left(\Lambda_x,\Lambda_z\right) &= \iint_{-\infty}^{\infty} \frac{e^{-i\left(\Lambda_x v_x + \Lambda_z v_z\right)} e^{-\left(v_x^2 + v_z^2\right)/2v_T^2}}{2\pi v_T^2} dv_x \, dv_z \equiv e^{-\left(\Lambda_x^2+\Lambda_z^2\right)\frac{v_T^2}{2}}
\end{aligned}\right\} \tag{62}$$

The first of equations (62) can be evaluated by using contour integration in the complex v_x plane. The integration over v_x gives

$$H_1 = -\frac{2\pi i L}{l\pi}\int_{-\infty}^{\infty} dv_z\, e^{i\Lambda_z v_z}\, e^{i\Lambda_x(\omega+i\nu)\frac{L}{l\pi}}\, e^{-i\Lambda_x\frac{k_o v_z L}{l\pi}\sin\theta} \qquad (\Lambda_x > 0)$$

$$= 0 \qquad (\Lambda_x < 0) \qquad (63)$$

The integration over v_z gives

$$H_1 = -(2\pi)^2\frac{iL}{l\pi}\, e^{i\frac{\Lambda_x\omega L\left(1+i\frac{\nu}{\omega}\right)}{l\pi}}\,\delta\left(\Lambda_z - \frac{\Lambda_x k_o L}{l\pi}\sin\theta\right) \qquad (\Lambda_x > 0)$$

$$= 0 \qquad (\Lambda_x < 0) \qquad (64)$$

Equation (61) therefore reduces to

$$J_l = -\left(1 + i\frac{\nu}{\omega}\right)\frac{v_T{}^2 L}{l\pi}\int_0^{\infty} d\Lambda_x\, e^{i\frac{\Lambda_x\omega L\left(1+i\frac{\nu}{\omega}\right)}{l\pi}}\, e^{-\left(1+\frac{k_o{}^2L^2\sin^2\theta}{l^2\pi^2}\right)\frac{\Lambda_x{}^2 v_T{}^2}{2}} \qquad (65)$$

and the other dispersion integrals become

$$J_{1l} = -\left(1 + i\frac{\nu}{\omega}\right)\frac{L}{l\pi}\, v_T{}^2\int_0^{\infty} d\Lambda_x\, e^{-\Lambda_x{}^2\frac{v_T{}^2}{2}\left(1+\frac{k_o{}^2L^2\sin^2\theta}{l^2\pi^2}\right)}\, e^{i\Lambda_x\frac{L}{l\pi}}\,\Lambda_x \qquad (66)$$

$$J_{2l} = \frac{iL}{l\pi}\, v_T{}^4\,\frac{k_o L}{l\pi}\sin\theta\int_0^{\infty} d\Lambda_x\, e^{i\Lambda_x\frac{\omega L}{l\pi}\left(1+i\frac{\nu}{\omega}\right)}\, e^{-\left(1+\frac{k_o{}^2L^2}{l^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x{}^2}{2}}\,\Lambda_x{}^2 \qquad (67)$$

$$J_{3l} = -\frac{\left(1 + i\frac{\nu}{\omega}\right)L}{l\pi}\,\frac{k_o L\sin\theta}{l\pi}\, v_T{}^2\int_0^{\infty} d\Lambda_x\, e^{i\Lambda_x\frac{\omega L}{l\pi}\left(1+i\frac{\nu}{\omega}\right)}\, e^{-\left(1+\frac{k_o{}^2L^2}{l^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x{}^2}{2}}\,\Lambda_x \qquad (68)$$

$$J_{4l} = -\frac{iL}{l\pi} v_T^2 \int_0^\infty d\Lambda_x \, e^{i\Lambda_x \frac{\omega L}{l\pi}\left(1+i\frac{\nu}{\omega}\right)} \, e^{-\left(1+\frac{k_o^2L^2}{l^2\pi^2}\sin^2\theta\right)\frac{\Lambda_x^2}{2}} \left(1 - \frac{\Lambda_x^2 k_o^2 L^2 v_T^2}{l^2\pi^2} \sin^2\theta\right) \tag{69}$$

As shown in appendix A, these integrals can further be reduced so that they can be expressed in terms of the dispersion function of Fried and Conte (ref. 23).

It should be emphasized again that G_1 and G_2 as previously given specify the surface impedance of the plasma, and therefore uniquely specify the reflection, transmission, and absorption coefficients for the plane wave problem. In appendix B, it is shown that results reduce to those of the half-space in the limit as $L \rightarrow \infty$, as they should.

As the ratio of v_T/v_{ph} (v_{ph} being the phase velocity of the wave) becomes small, the imaginary parts of the dispersion integrals J_{ln} are negligible, and the real parts of the dispersion integrals can be expanded in increasing powers of v_T/v_{ph}, as shown in appendix A. The results, when applied to equations (68) and (69) are

$$G_1 \approx \left(\frac{1}{k_oL}\right)^2 \sum_{l=0}^{\infty} \frac{2}{1+\delta_o{}^l} \frac{1 - \dfrac{\sin^2\theta}{1 - \dfrac{\omega_p{}^2}{\omega^2} - 3\dfrac{\omega_p{}^2}{\omega^2}\dfrac{v_T{}^2}{c^2}\left(\dfrac{l\pi}{k_oL}\right)^2}}{1 - \dfrac{\omega_p{}^2}{\omega^2} - \sin^2\theta - \left(\dfrac{l\pi}{k_oL}\right)^2} \tag{70}$$

$$G_2 \approx \left(\frac{1}{k_oL}\right)^2 \sum_{l=0}^{\infty} \frac{2(-1)^l}{1+\delta_o{}^l} \frac{1 - \dfrac{\sin^2\theta}{1 - \dfrac{\omega_p{}^2}{\omega^2} - 3\dfrac{\omega_p{}^2}{\omega^2}\dfrac{v_T{}^2}{c^2}\left(\dfrac{l\pi}{k_oL}\right)^2}}{1 - \dfrac{\omega_p{}^2}{\omega^2} - \sin^2\theta - \left(\dfrac{l\pi}{k_oL}\right)^2} \tag{71}$$

Resonances occur in G_1 and G_2 when

$$l\pi = \frac{k_oL}{\sqrt{3}\dfrac{v_T}{c}} \frac{\sqrt{1 - \dfrac{\omega_p{}^2}{\omega^2}}}{\dfrac{\omega_p}{\omega}} \tag{72}$$

which is approximately the same as the hydrodynamic results for $\omega_p/\omega \approx 1$. With κ defined as $l\pi/L$, it is noted that the resonances defined by equation (72) occur when

$$\frac{v_{ph}}{v_T} = \frac{\omega}{\kappa v_T} = \frac{k_o L}{l\pi \frac{v_T}{c}} \tag{73}$$

Thus the kinetic effect, that is, Landau damping, should become more pronounced as the ratio of v_{ph}/v_T becomes smaller. Therefore, for fixed v_T/c and fixed k_oL, the Landau damping should become more severe as the order of the resonance increases. Numerical results of the reflection and absorption coefficients are discussed subsequently.

Kinetic Results for Electric Field Perpendicular to Plane of Incidence

It has been indicated that most of the interesting effects associated with a plane wave obliquely incident upon a plasma layer occur with longitudinal plasma waves excited in the plasma. As such, the case of the electric vector perpendicular to the plane of incidence $\left(\vec{E} = E_y\vec{u}_y \text{ in fig. } 2\right)$ is only of secondary interest. However, the results are included here for completeness. By proceeding as in the section "Direct Solution of Plane Wave Problem Using Linearized Vlasov Equation," the Fourier coefficients of F^- are found to be

$$F_l^- = \frac{-2i\omega_p^2}{e}\frac{k_o^2}{\omega}\frac{F_o v_y v_x H_{lz}}{\left(\omega - k_o v_z \sin\theta + i\nu\right)^2 - \left(\frac{l\pi v_x}{L}\right)^2} \tag{74}$$

It further follows from the Maxwell curl equations that

$$\nabla^2 H_z + k_o^2 H_z = -\frac{\partial j_y}{\partial x} \tag{75}$$

or

$$\frac{\partial^2 H_z}{\partial x^2} + \left(k_o^2 - k_o^2 \sin^2\theta\right)H_z = -e\int d\vec{v}\, v_y \frac{\partial F_1^+}{\partial x}$$

$$= -ie\int d\vec{v}\, \frac{v_y}{v_x}\left(\omega - k_o v_z \sin\theta\right)F_1^- \tag{76}$$

A Fourier expansion solution of equation (76) gives

$$H_z = \sum_{l=1}^{\infty} \frac{k_o \frac{l\pi}{L} \frac{H_z(0) - (-1)^l H_z(L)}{L/2}}{1 - \sin^2\theta - \left(\frac{l\pi}{k_o L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_l} \tag{77}$$

where

$$J_l = \iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_o}{\omega - k_o v_z \sin\theta - \frac{l\pi v_x}{L} + i\nu}$$

$$\equiv -\frac{iL}{l\pi} \int_0^{\infty} d\Lambda_x \, e^{-\Lambda_x^2 \frac{v_T^2}{2}\left(1 + \frac{k_o^2 L^2}{l^2\pi^2} \sin^2\theta\right)} e^{i\Lambda_x \frac{\omega L}{l\pi}\left(1 + i\frac{\nu}{\omega}\right)} \tag{78}$$

If $R^{\perp}$ and $T^{\perp}$ are defined as the reflection and transmission coefficients for the case where the electric field is perpendicular to the plane of incidence, then

$$E_y = E_o\left(e^{ik_o x \cos\theta} + R^{\perp} e^{-ik_o x \cos\theta}\right) e^{i(k_o z \sin\theta - \omega t)}$$

for $x < 0$ and

$$E_y = E_o T^{\perp} e^{i(k_o x \cos\theta + k_o z \sin\theta - \omega t)}$$

for $x > L$. If the boundary-value problem is completed, it is found that

$$\frac{1 + R^{\perp}}{1 - R^{\perp}} = Z = -ik_o L \cos\theta \left[G_1^{\perp} - \frac{\left(G_2^{\perp}\right)^2}{G_1^{\perp} + \frac{i}{k_o L \cos\theta}}\right] \tag{79}$$

where Z is the surface impedance, and the functions $G_1^{\perp}$ and $G_2^{\perp}$ are given by

$$G_1^{\perp} = \left(\frac{1}{k_o L}\right)^2 \sum_{l=0}^{\infty} \frac{2}{1 + \delta_o^{\,l}} \frac{1}{1 - \sin^2\theta - \left(\frac{l\pi}{k_o L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_l} \tag{80}$$

$$G_2^{\perp} = \left(\frac{1}{k_o L}\right)^2 \sum_{l=0}^{\infty} \frac{2}{1 + \delta_o^{\,l}} \frac{(-1)^l}{1 - \sin^2\theta - \left(\frac{l\pi}{k_o L}\right)^2 - \omega \frac{\omega_p^2}{\omega^2} J_l} \tag{81}$$

Equation (79) is slightly different in form from that previously obtained for the case of parallel incidence; this difference occurs because $R^{\perp}$, in this case, is the ratio of the reflected to the incident electric field.

Relationship to Plasma Capacitor Problem

It has already been pointed out that although the ratio of v_T/v_{ph} is ordinarily negligible for transverse waves, parallel-polarized waves might be expected to show interesting kinetic effects, as shown. As further support of this assumption, it is useful to compare the plane-wave calculations with corresponding computations in which only longitudinal plasma oscillations are excited. The plasma capacitor provides this information.

The problem under consideration consists of a plane-parallel plasma-filled capacitor, whose plates are located at $x = 0$ and $x = L$. An electric field, oscillating at an angular frequency ω, is applied normal to the plates. In this section it is shown that the capacitor exhibits resonance behavior at the same values of $k_o L$, $(\omega_p/\omega)^2$, and v_T/v_{ph}. In order to effect this behavior, the impedance of the capacitor will be determined, as done by Hall (ref. 18) and Shure (ref. 24).

If E_z and ν are set equal to zero in equations (40) and (41), and a process analogous to that which led to the expression for the x-component of the current density (eq. (46)) is followed, the Fourier coefficients of the current density in the capacitor are found to be

$$j_{lx} = \frac{i\omega_p^2 \omega \epsilon_o}{v_T^2} \frac{L}{l\pi} J_{1l} E_{lx} \tag{82}$$

which is nothing more than equation (46) with $H_{ly} = 0$. The continuity equation, relating charge density and current density, gives

$$-\frac{I}{A} = j_x - i\omega\epsilon_o E_x = \text{Constant} \tag{83}$$

where I/A is the current per unit area on the plate of the capacitor. A Fourier expansion of equation (83) gives

$$\frac{I}{A}\frac{2}{L}\frac{1-(-1)^l}{l\pi/L} = i\omega\epsilon_o\left(1 - \frac{\omega_p^2}{v_T^2}\frac{L}{l\pi}J_{1l}\right)E_{lx} \tag{84}$$

Since the voltage between the plates is given by $V = -\int_0^L E_x\,dx$, the impedance of the capacitor is given by

$$Z = -\frac{1}{i\omega C}\left(\frac{2}{\pi}\right)^2 \sum_{l=1}^{\infty} \frac{1-(-1)^l}{l^2\left(1 - \frac{\omega_p^2}{v_T^2}\frac{L}{l\pi}J_{1l}\right)} \tag{85}$$

where C is the capacitance in the absence of a plasma. If $\Lambda^+ = 1 - \frac{\omega_p^2}{v_T^2}\frac{L}{l\pi}J_{1l}$, equation (85) becomes identical to that obtained by Shure (ref. 24). For $v_T/v_{ph} \ll 1$, equation (85) reduces to

$$Z \approx \frac{1}{i\omega C}\left(\frac{2}{\pi}\right)^2 \sum_{l=1}^{\infty} \frac{1-(-1)^l}{l^2\left[1 - \frac{\omega_p^2}{\omega^2} - 3\frac{\omega_p^2}{\omega^2}\left(\frac{v_T l\pi}{\omega L}\right)^2\right]} \tag{86}$$

An inspection of the denominator of the sum in equation (86) shows that resonances occur when

$$l\pi = \frac{k_o L}{\sqrt{3}\,\frac{v_T}{c}} \frac{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{\frac{\omega_p}{\omega}} \tag{87}$$

Equation (87) is identical to equation (72), which defined the resonance condition for a plane wave incident upon the slab. Therefore similarities between the plane wave and capacitor results should be expected. These similarities are exhibited in the form of numerical calculations in the next section.

Numerical Results

In order to delineate clearly the differences between using cold, fluid, and kinetic models for a plane electromagnetic wave obliquely incident upon a plasma layer, computations of the reflection coefficient for each model were made as a function of $(\omega_p/\omega)^2$ with the following parameters fixed:

$$\theta = 15^{o}$$

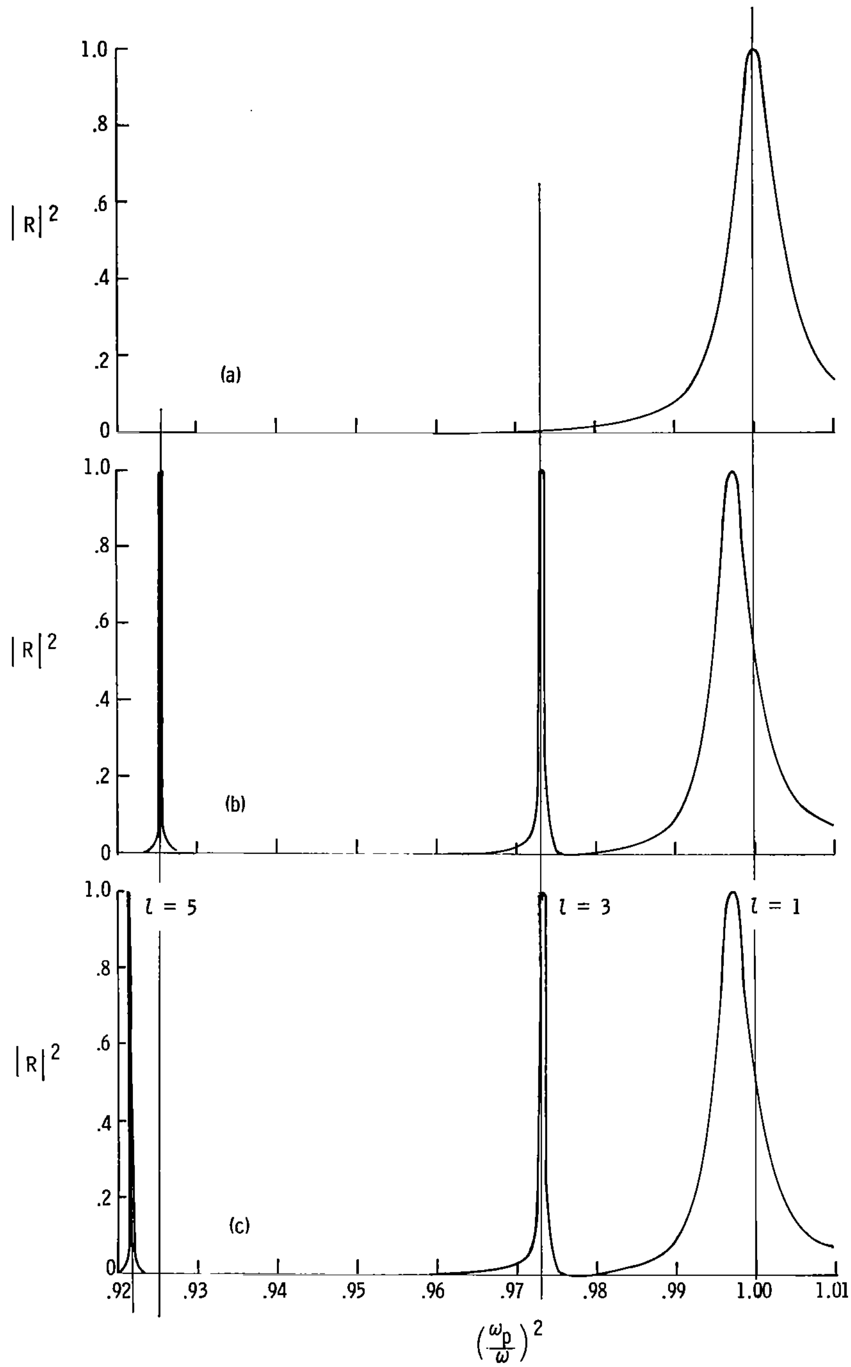
$$k_oL = 0.1$$

$$\frac{\nu}{\omega} = 0$$

$$\frac{v_T}{c} = 10^{-3}$$

and $v_T/v_{ph} = 0.157$. The results are shown in figure 3. Note that for the cold plasma (fig. 3(a)), only one resonance occurs, and that is located at $\omega = \omega_p$. In the fluid limit (fig. 3(b)), a series of resonances occur, corresponding to the zeros of $\sin\left(\frac{\omega}{a}\sqrt{1 - \frac{\omega_p{}^2}{\omega^2}}L\right)$. Also note that the main resonance is displaced from $(\omega_p/\omega)^2 = 1$ and that each higher order resonance is narrower than its predecessor. The results obtained from the Vlasov equation are similar to those obtained with the hydrodynamic equations, except that the resonances do not occur at the same values of $(\omega_p/\omega)^2$. The departure becomes more pronounced with increasing order, but this difference is to be expected since the fluid approximation becomes less valid. (For example, compare eqs. (37) and (72).) Also note that a resonance is associated with each odd term of the Fourier expansion.

When the phase velocity of the wave becomes of the same order as the thermal velocity of the plasma, electrostatic energy in the wave is absorbed by the electrons and converted to kinetic energy. This condition occurs because of the inertial "drag" of the electrons, and is the well-known phenomenon of Landau damping. In order to demonstrate the effects of Landau damping, the $l = 5$ resonance was investigated for additional values of the parameters. The reflection and absorption coefficients are shown in figure 4 for various values of v_T/v_{ph}, where the abscissa has been grossly exaggerated. For $v_T/v_{ph} = 0.157$, little Landau damping occurs, as evidenced by a narrow resonance having a peak reflection coefficient near unity. However, absorption becomes very pronounced with only a small increase in v_T/v_{ph}. Note that the resonance is heavily Landau damped at a ratio of v_T/v_{ph} of less than 0.2. It is somewhat surprising



(a) Cold plasma theory.

(b) Linearized fluid model.

(c) Model based on the linearized Vlasov equation.

Figure 3.- Reflection coefficient of a plasma wave incident upon a plasma slab. $k_0L = 0.1$; $v_T/c = 10^{-3}$; $v_T/v_{ph} = 0.157$.

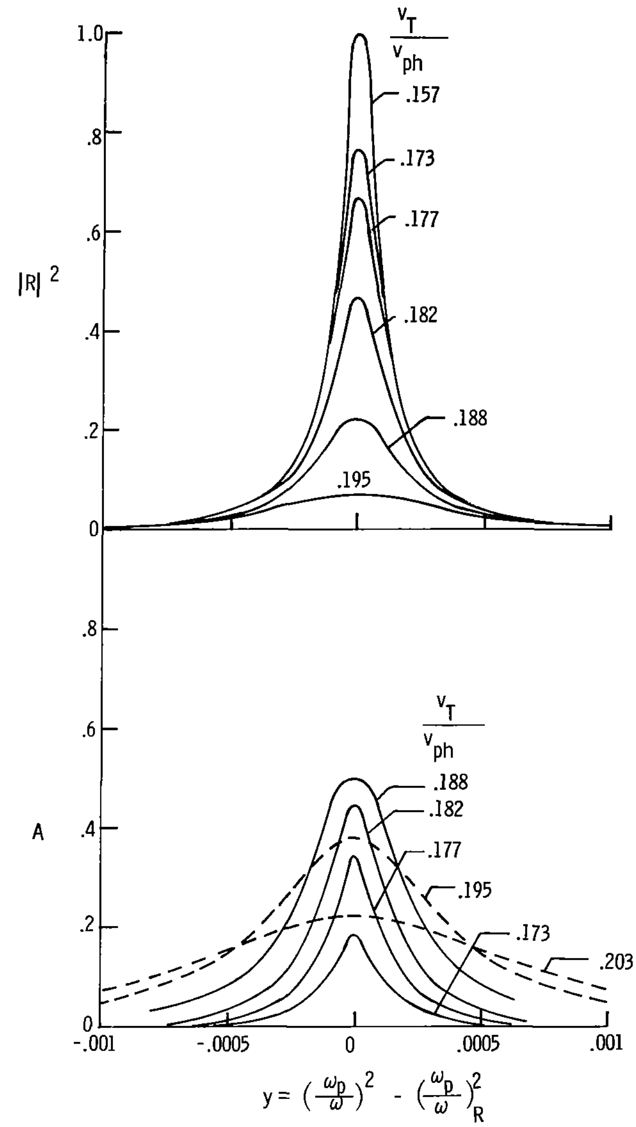


Figure 4.- Reflection and absorption coefficients as a function of the plasma electron density for various electron thermal velocities. $l = 5$; $k_0L = 0.1$.

to see so much damping at such a low value of v_T/v_{ph}, but this condition occurs because the width of the resonance is so small. The pertinence of the line width is discussed later when collisional damping is considered. The resonant peak also occurs at smaller values of $\left(\omega_p/\omega\right)^2$ as v_T/v_{ph} increases, as is evidenced in figure 5, where the abscissa is again exaggerated. The reflection and absorption coefficients are plotted as functions of $\left(\omega_p/\omega\right)^2$, for various values of the angle of incidence, in figure 6 with the following parameters fixed:

$$k_o L = 0.1$$

$$\frac{\nu}{\omega} = 0$$

$$\frac{v_T}{c} = 1.16 \times 10^{-3}$$

and $v_T/v_{ph} = 0.182.$

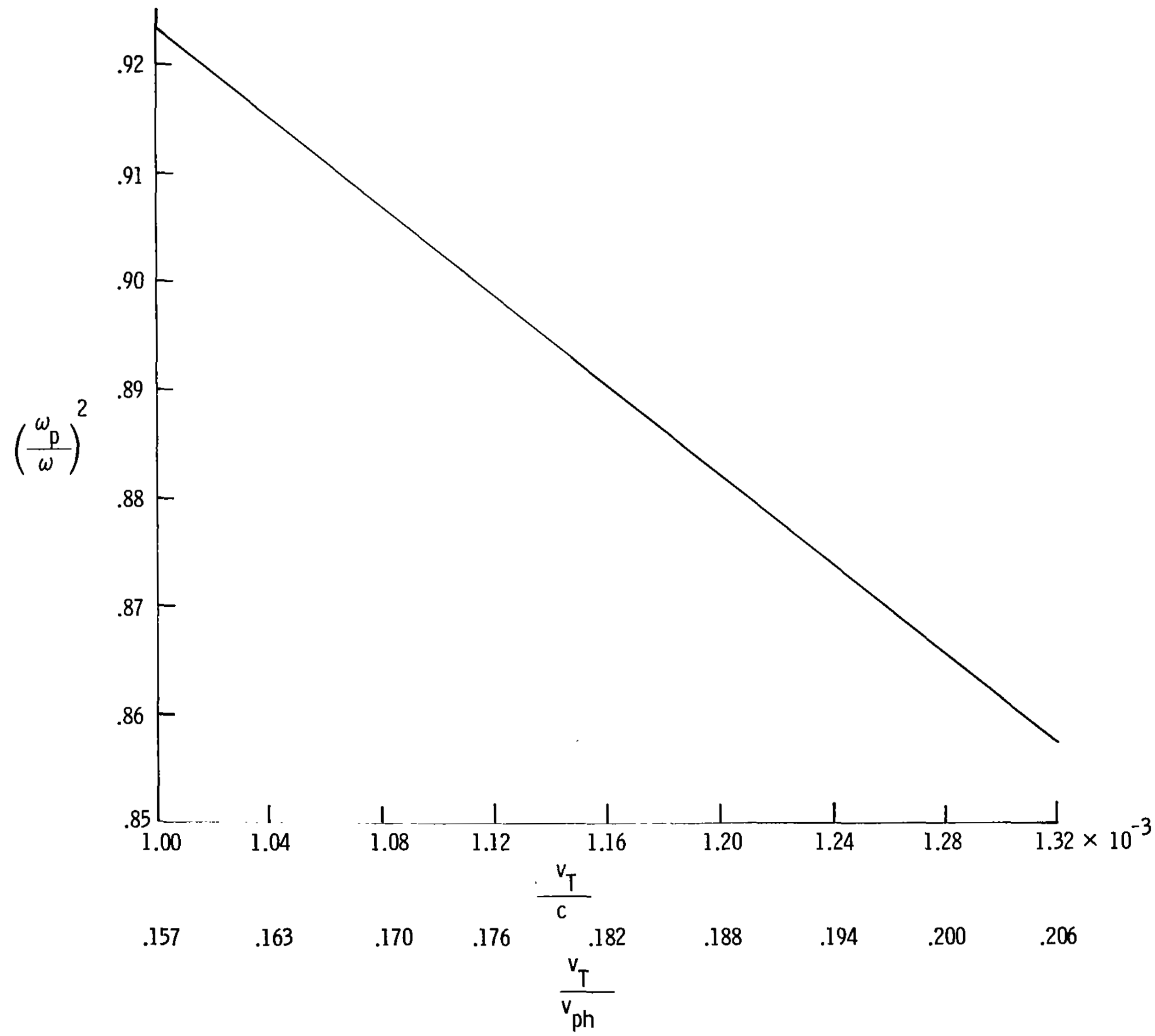


Figure 5.- Resonant frequency as a function of electron density and electron thermal velocity. $l = 5$; $k_o L = 0.1$.

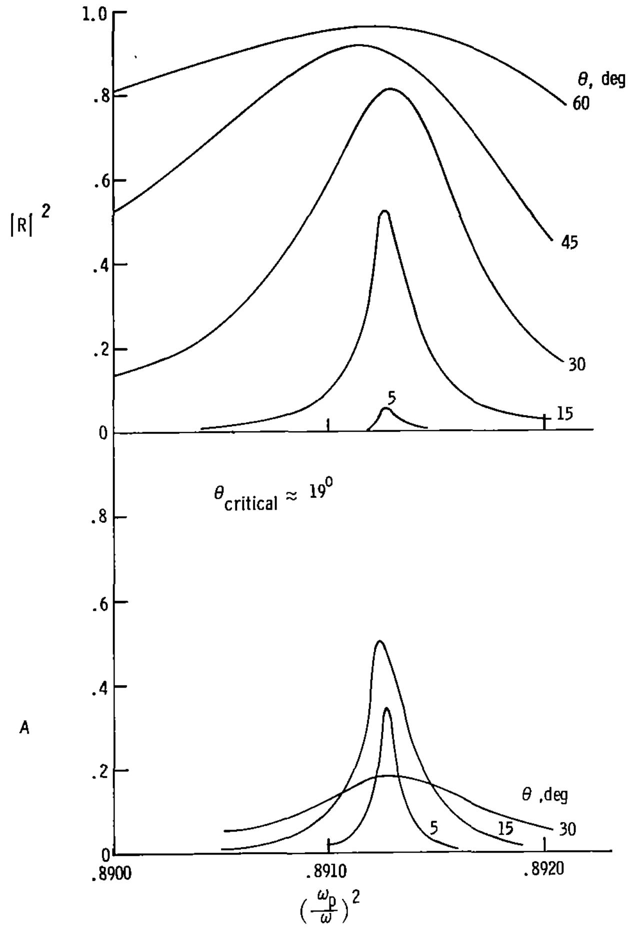


Figure 6.- Reflection and absorption coefficients as a function of electron density for various angles of incidence. $l = 5$; $k_0L = 0.1$; $v_T/c = 1.16 \times 10^{-3}$; $v_T/v_{ph} = 0.182$.

As the angle of incidence increases from $\theta = 5^{\circ}$ to $\theta = 15^{\circ}$, the peak value of the absorption coefficient increases, in large part, because the longitudinal component of the electric field increases in proportion to $\sin \theta$. As θ further increases, the transverse electromagnetic waves become evanescent within the slab. Since the transverse waves and longitudinal waves are coupled, this condition leads to a damping of the resonance.

Since electromagnetic waves in a plasma propagate as $e^{\pm ik_p x} = e^{\pm ik_o x\sqrt{1-\frac{\omega_p^2}{\omega^2}-\sin^2\theta}}$, the waves become evanescent when $\theta = \cos^{-1}\frac{\omega_p}{\omega}$, which is about 19° for the case considered here. Furthermore, note in figure 6 that there is very little shift in the position of the resonance with increasing values of the angle of incidence.

The reflection and absorption coefficients plotted as a function of $\left(\omega_p/\omega\right)^2$ and ν/ω are shown in figure 7 for the model based on the Vlasov equation, with the following parameters fixed:

$$\theta = 15^{o}$$

$$k_oL = 0.1$$

$$\frac{v_T}{c} = 10^{-3}$$

and $v_T/v_{ph} = 0.157$.

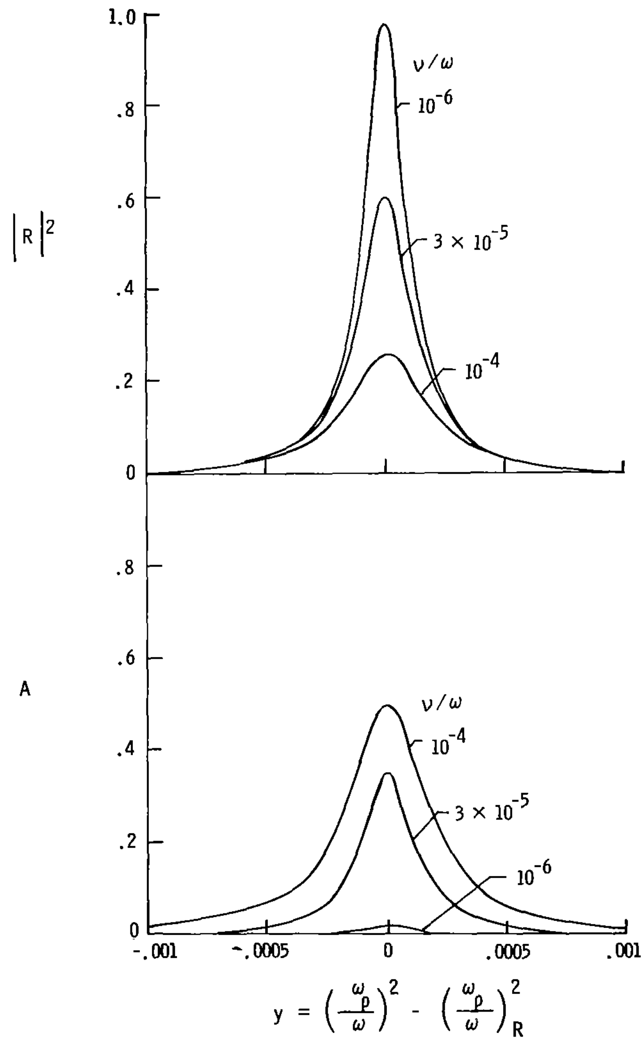


Figure 7.- Reflection and absorption coefficients as a function of electron density for various values of collision frequency for a plasma model based on the linearized Vlasov equation with a Bhatnagar-Gross-Krook collision term. $l = 5$; $k_oL = 0.1$; $v_T/c = 10^{-3}$; $v_T/v_{ph} = 0.157$.

For values of ν/ω less than about 10^{-6}, the collisions do not appreciably influence the reflection and absorption coefficients; however, as ν/ω increases a couple of orders of magnitude, the damping becomes pronounced. Although it may, at first glance, seem surprising that such a large effect occurs for ν/ω as low as 10^{-4}, from figure 7 it can be seen that the line width is of the order of 10^{-4}. From our general knowledge of resonance phenomena, damping is expected to be appreciable whenever the collision frequency is of the order of the line width. This condition also accounts for the degree of Landau damping observed in figure 4. Similar conclusions can be drawn by inspecting the fluid results shown in figure 8.

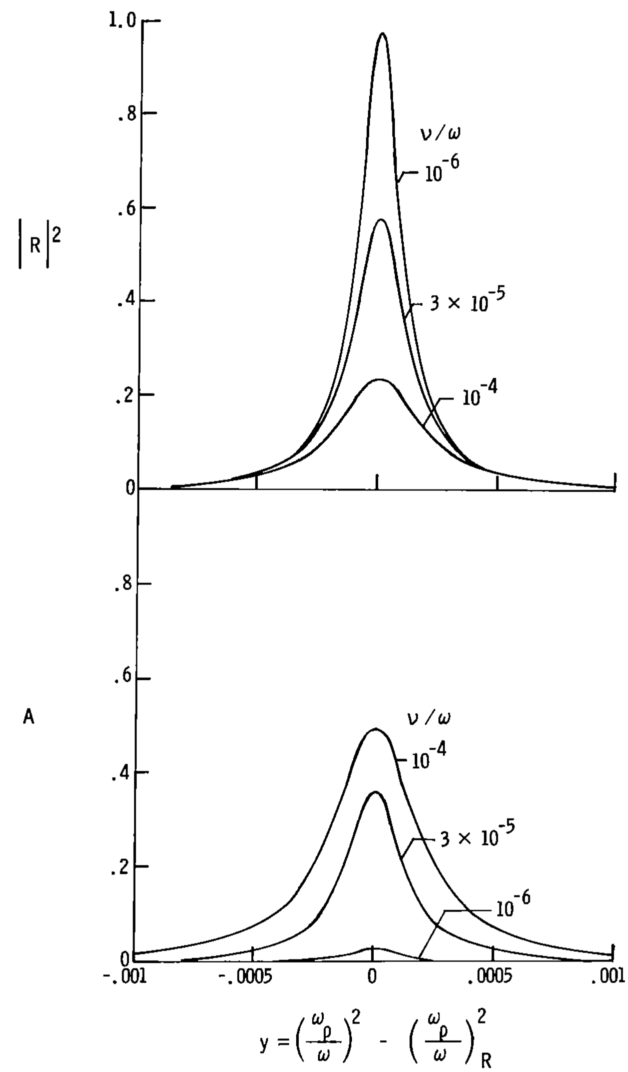
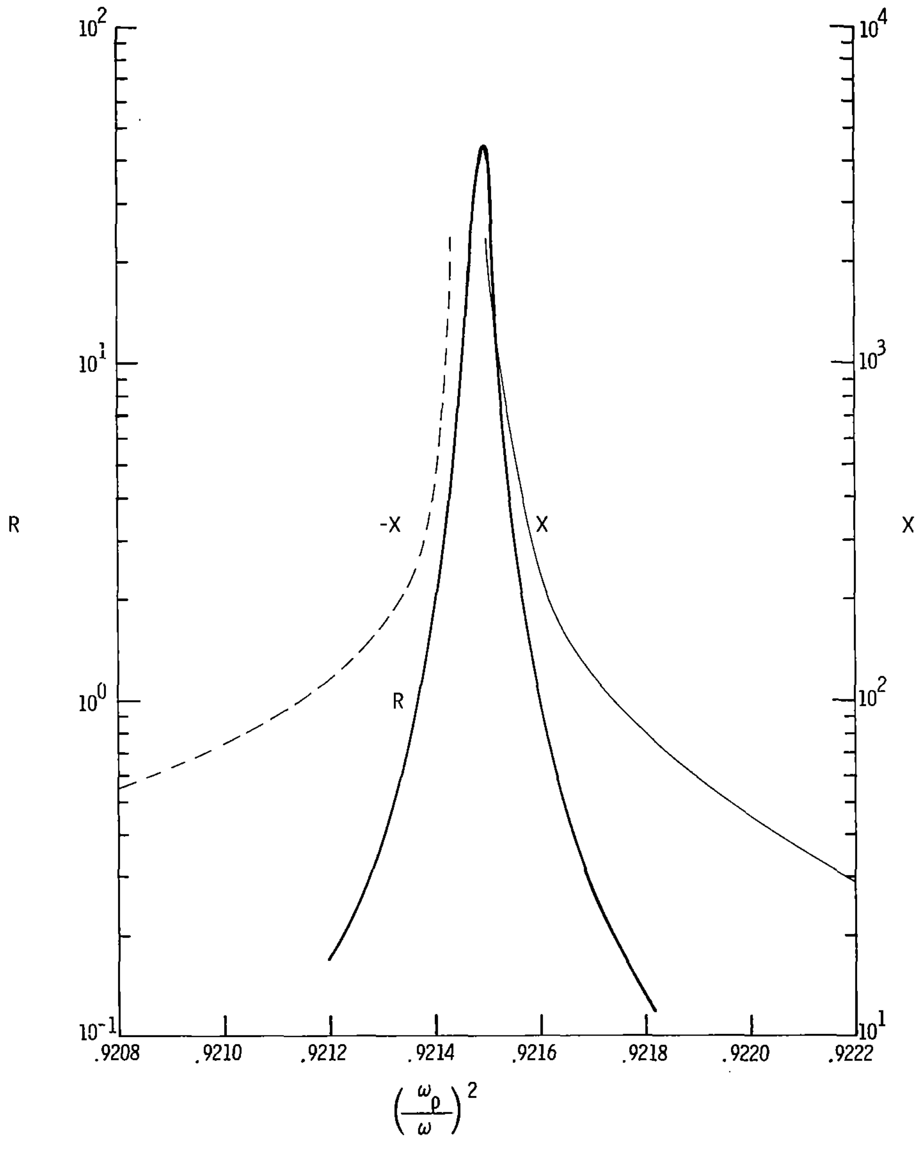


Figure 8.- Reflection and absorption coefficient as a function of electron density with collision frequency as a parameter for a linearized fluid equation. $l = 5$; $k_0L = 0.1$; $v_T/c = 10^{-3}$; $v_T/v_{ph} = 0.157$.

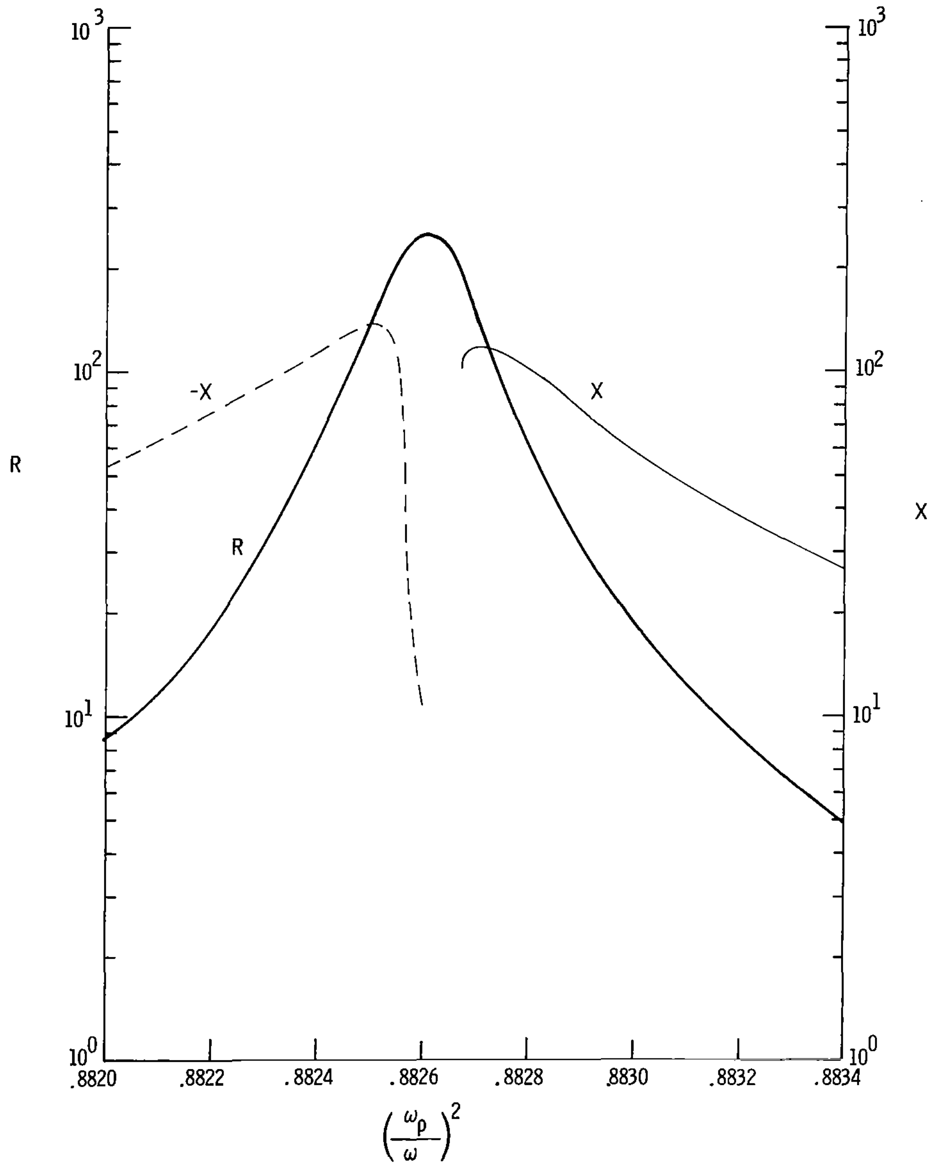
Computer results for the impedance of the plasma capacitor are shown in figure 9, when the resistance and reactance (normalized to $X_o = 1/\omega C$) are plotted as a function of $(\omega_p/\omega)^2$. Figure 9(a) gives results for $v_T/c = 10^{-3}$ $(v_T/v_{ph} = 0.157)$; figure 9(b), for $v_T/c = 1.20 \times 10^{-3}$ $(v_T/v_{ph} = 0.188)$; and figure 9(c), for $v_T/c = 1.30 \times 10^{-3}$ $(v_T/v_{ph} = 0.203)$. In each case, the normalized slab thickness k_oL is fixed at 0.1. Therefore, except for the angle of incidence, which does not appear in the capacitor



(a) $v_T/c = 10^{-3}$; $v_T/v_{ph} = 0.157$.

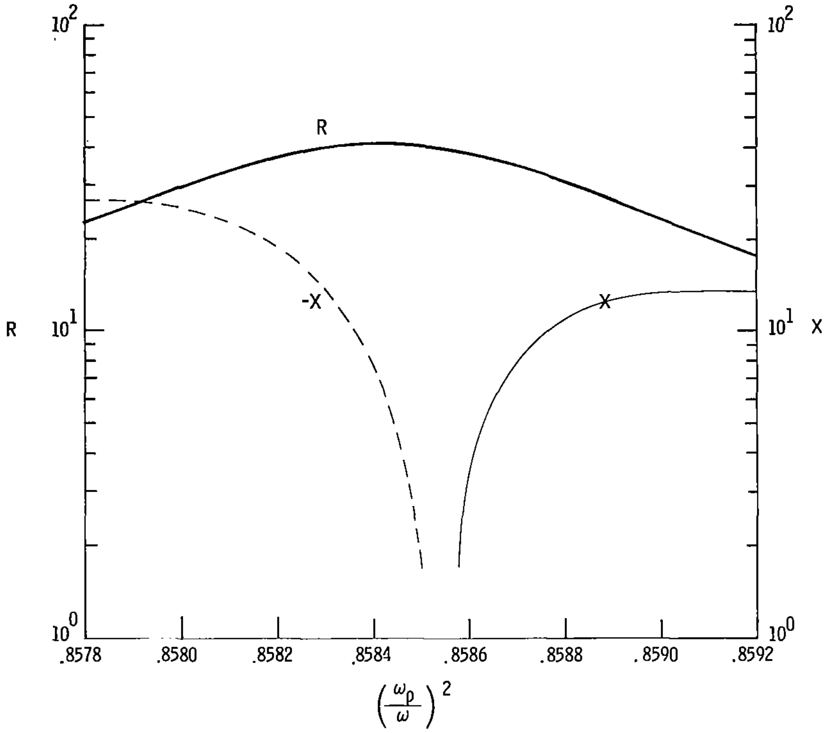
Figure 9.- The impedance of a plasma capacitor as a function of electron density for several values of the thermal velocity. $l = 5$; $k_oL = 0.1$.

expression, all pertinent capacitor parameters are the same as those for the $l = 5$ plane-wave resonance. From figure 9 it can be seen that the resonance is very sharp for low values of v_T/c, and the width at half-maximum noticeably broadens as the thermal velocity increases, as does the reflection coefficient of the plane wave. Note further that if figure 9 is compared with figures 4 and 5, the peak of the resonance for the capacitor and the obliquely incident parallel-polarized plane wave occurs at precisely the same values of the plasma parameters.



(b) $v_T/c = 1.20 \times 10^{-3}$; $v_T/v_{ph} = 0.188$.

Figure 9.- Continued.



(c) $v_T/c = 1.30 \times 10^{-3}$; $v_T/v_{ph} = 0.203$.

Figure 9.- Concluded.

RESULTS AND DISCUSSION

The prime objective in the work described herein was to examine analytically and numerically the details of the coupling phenomena that occurs between electromagnetic waves and longitudinal plasma oscillations. This examination was made in order to relate this problem to the Tonks-Dattner resonances, which are known to occur when the electromagnetic field is applied to an inhomogeneous plasma in such a way as to couple to longitudinal plasma waves. The model of the inhomogeneous plasma used was of the simplest type, a thin uniform plasma slab. The specific boundary-value problem considered an electromagnetic plane wave obliquely incident upon the slab, and assumed specular reflection of plasma electrons from the faces of the slab. By solving for the reflection, transmission, and absorption coefficients, the detailed behavior of the resonances, that is, the shift in resonant frequency and changes in the width at half-maximum of the resonances in the reflection coefficient as a function of the plasma parameters could be examined. The problem was approached in such a way as to delineate the differences, both from a physical and computational viewpoint, between a cold plasma model, a fluid model, and a model based on the Vlasov equation. The case where the electric vector was polarized parallel to the plane of incidence was considered in detail. Only this polarization excites longitudinal plasma oscillation, for which the ratio of the thermal velocity of the particles to the phase velocity of the wave is not so small as to diminish the kinetic effects. The analysis further showed that from the practical viewpoint, only the lower order longitudinal plasma resonances (that is, $l = 1$, 3, and 5) can be supported; these resonances are well separated only if the plasma slab is thin compared with the wavelength of the incident wave. If the plasma is thick, the resonances become more grouped near $\omega = \omega_p$, for other parameters fixed, as evidenced from equation (72). The cold plasma model predicts for such a slab that only one resonance can be supported, and that resonance occurs when the signal frequency equals the plasma frequency. The fluid model was found to support a series of resonances; these resonances became narrower as the order increased, and are similar in nature to the Tonks-Dattner resonances in the sense that the secondary resonances occur at successively lower values of the electron density than the main resonance does. Similar resonances were observed when a kinetic analysis using the Vlasov equation was undertaken except that electron densities for resonance were shifted, and the effects of Landau damping became evident as the thermal velocity increased. This result is a manifestation of the fact that the ratio v_T/v_{ph} can no longer be considered negligible as the order increases. This damping with increasing order is qualitatively consistent with the experimental observations of the behavior of the Tonks-Dattner resonances. Such collisionless damping cannot be anticipated from the fluid equations. The fluid equations, however, have the advantage of presenting a simple physical picture of the standing wave processes that occur within the slab.

The pertinence of the parallel-plate plasma-filled capacitor problem was demonstrated by calculating the impedance of the capacitor as a function of slab thickness, plasma frequency, propagating frequency, and thermal velocity. It was found that when the slab is driven either by the capacitor or by a plane wave, resonances occur at the same values of these parameters, and that Landau damping commences in both problems at the same values of the thermal velocity.

Most of the computational effort pertaining to the Vlasov equation consisted of evaluating the dispersion function, and using it to determine numerically the series given by equations (68) and (69). (See appendix C for details.) This series would, in turn, be used to determine reflection, transmission, and absorption coefficients. One of the more surprising aspects of the computations was that, for the cases considered, the infinite series given by equations (68) and (69) can accurately be represented by only two terms in the series. These terms consist of the first ($l = 0$) term and that odd term ($l = 1,3,5,\ldots$) which corresponds most nearly to resonance for the parameters under consideration. This result is to be contrasted with the half-space solution (to which the results reduce as the slab dimension goes to infinity) which usually requires more involved computational techniques.

The influence of collisional damping of the resonances is then considered. For the particular parameters considered, it was observed that unless the ratio of collision frequency to signal frequency is less than about 10^{-4}, the third odd resonance is damped out. Since a collision frequency ratio of less than 10^{-4} for gaseous plasmas is not easy to achieve in the laboratory, it is not difficult to conclude that the uniform slab can support the experimentally observed Tonks-Dattner resonances.

It is a temptation to conclude that the inhomogeneity of the plasma does more than merely control the spacing of the resonances (ref. 7). The results lead us to believe that the inhomogeneity may also tend to broaden the width of the higher order resonances at half-maximum, which would make them less sensitive to collisional and Landau damping. This broadening could occur either because the inhomogeneity provides a gradual transition in the impedance between the plasma and the air interface, or because the inhomogeneity induces a background field which causes resonant trapping of the electrons. An investigation of the latter problem (that is, consideration of the background field) is a very formidable task, indeed. The description of even the simplest problems has developed into enormous and frustrating computer programing projects (refs. 25 and 26), and illuminating results are difficult to achieve unless many simplifying assumptions are made (ref. 27). However, such results should present some new and rather interesting kinetic effects, and the problem is therefore worthwhile to pursue. If the background field can be ignored (ref. 28), the problem becomes more tractable (but still numerically much more difficult than that for the uniform plasma). Such a solution may be valuable in order to determine whether a gradual reduction in electron density at the boundaries will broaden the resonances.

The techniques described here can easily be extended to other problems such as: (1) the study of the impedance characteristics of antennas under plasmas (ref. 9), (2) the effects of nonspecular reflecting boundaries, and (3) electromagnetic waves obliquely incident upon a plasma slab in which a static magnetic field is applied normal to the boundaries. All these problems are of interest and may be approached by extending the techniques of the current work.

CONCLUDING REMARKS

The classical problem of an electromagnetic wave obliquely incident upon a plasma slab was investigated by using the coupled Maxwell-Vlasov equations. For the parameters considered, a plot of reflection coefficient against electron density reveals a series of resonances which become narrower and Landau damped as the order of the resonance increases. This behavior is in qualitative agreement with experimentally observed Tonks-Dattner resonances. However, on a quantitative basis, the computed resonances are much closer together than experiment indicates; this effect is probably due to the inhomogeneity of the plasma.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., January 6, 1970.

APPENDIX A

FURTHER PROPERTIES OF THE DISPERSION INTEGRALS

If the collision frequency ν is zero, the various dispersion integrals may be defined as follows:

$$\begin{bmatrix} J_l \\ J_{1l} \\ J_{2l} \\ J_{3l} \\ J_{4l} \\ J_{5l} \end{bmatrix} = \iint_{-\infty}^{\infty} \frac{dv_x \, dv_z \, F_o}{\omega - k_o v_z \sin \theta - \frac{l\pi v_x}{L}} \begin{bmatrix} 1 \\ v_x \\ v_x v_z \\ v_z \\ v_z^2 \\ v_x^2 \end{bmatrix} \tag{A1}$$

where the integration over v_y has been performed to give

$$\int_{-\infty}^{\infty} dv_y \, F_o(v_x, v_y, v_z) = F_o(v_x, v_z)$$

which is the two-dimensional Maxwellian distribution function. From the integral relationships,

$$\iint_{-\infty}^{\infty} v_x F_o \, dv_x \, dv_z = \iint_{-\infty}^{\infty} v_z F_o \, dv_x \, dv_z = 0 \tag{A2}$$

and

$$\iint_{-\infty}^{\infty} F_o \, dv_x \, dv_z = 1 \tag{A3}$$

It is possible to show that by multiplying the numerators and denominators of the integrands of equations (A2) and (A3) by $\omega - k_z v_z \sin \theta - \frac{l\pi v_x}{L}$,

$$J_{1l} = \frac{J_{2l}}{c} \sin \theta + \frac{l\pi}{\omega L} J_{5l} \tag{A4}$$

$$J_{3l} = \frac{J_{4l}}{c}\sin\theta + \frac{l\pi}{\omega L}J_{2l} \tag{A5}$$

$$1 = \omega J_l - \left(\frac{l\pi}{L}J_{1l} + k_o \sin\theta\, J_{3l}\right) \tag{A6}$$

$$J_{3l} = \frac{k_o L}{l\pi}\sin\theta\, J_{1l} \tag{A7}$$

As shown in the text, it is possible to use Fourier transforms in velocity space to re-express equation (A1) in terms of single integrals over the transform variable Λ_x. The results are:

$$\begin{bmatrix} J_l \\ J_{1l} \\ J_{2l} \\ J_{3l} \\ J_{4l} \\ J_{5l} \end{bmatrix} = -\frac{iL}{l\pi}\int_0^\infty d\Lambda_x\, e^{-\Lambda_x^2 \frac{v_T^2}{2}\left(1+\frac{k_o^2 L^2 \sin^2\theta}{l^2\pi^2}\right)} e^{i\frac{\Lambda_x L}{l\pi}} \begin{bmatrix} 1 \\ -i\Lambda_x v_T^2 \\ -v_T^4 \Lambda_x^2 \frac{k_o L}{l\pi}\sin\theta \\ -i\Lambda_x \frac{k_o L v_T^2}{l\pi}\sin\theta \\ v_T^2\left(1 - \frac{\Lambda_x^2 k_o^2 L^2 v_T^2}{l^2\pi^2}\right)\sin^2\theta \\ v_T^2\left(1 - \Lambda_x^2 v_T^2\right) \end{bmatrix} \tag{A8}$$

In the limit as v_T approaches 0, the exponential involving v_T^2 may be expanded as a Taylor series to give approximations for the real parts of the dispersion integrals as the thermal velocity of the plasma approaches zero. The results are

$$J_l \approx \frac{1}{\omega}\left[1 + \frac{v_T^2}{\omega^2}\left(k_o^2 \sin^2\theta + \frac{l^2\pi^2}{L^2}\right)\right] \tag{A9}$$

$$J_{1l} \approx \frac{v_T^2}{\omega^2}\frac{l\pi}{L}\left[1 + \frac{3v_T^2}{\omega^2}\left(\frac{l^2\pi^2}{L^2} + k_o^2 \sin^2\theta\right)\right] \tag{A10}$$

$$J_{2l} \approx 2k_o \sin\theta \frac{l\pi}{L} \frac{v_T^4}{\omega^3} \tag{A11}$$

$$J_{3l} \approx \frac{k_o L}{l\pi} \sin\theta \, J_{2l} \tag{A12}$$

$$J_{4l} \approx \frac{v_T^2}{\omega}\left[1 + \frac{v_T^2}{\omega^2}\left(\frac{l^2\pi^2}{L^2} + 3k_o^2 \sin^2\theta\right)\right] \tag{A13}$$

$$J_{5l} \approx \frac{v_T^2}{\omega}\left[1 + \frac{v_T^2}{\omega^2}\left(\frac{3l^2\pi^2}{L^2} + k_o^2 \sin^2\theta\right)\right] \tag{A14}$$

These integrals are now related to the tabulated plasma dispersion function, as given by Fried and Conte (ref. 23). Fried and Conte define the dispersion function as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx \, e^{-x^2}}{x - \zeta} \tag{A15}$$

which may be recognized as the Hilbert transform of the Gaussian. If a change of variables is used so that $x = \frac{v_x}{\sqrt{2}a}$, with

$$a = v_T \sqrt{1 + \frac{k_o^2 L^2 \sin^2\theta}{l^2\pi^2}}$$

and ζ is defined as

$$\zeta = \frac{\omega L}{l\pi} \frac{1}{\sqrt{2} v_T} \frac{1}{\sqrt{1 + \frac{k_o^2 L^2 \sin^2\theta}{l^2\pi^2}}}$$

the velocity transforms may be used to show that

$$Z(\zeta) = -\frac{\omega}{\zeta} J_l(\zeta) \tag{A16}$$

Further manipulations may be performed to show that

$$J_{1l}(\zeta) = -\frac{2l\pi}{k_oL}\left(\frac{v_T}{c}\right)^2 \frac{1}{k_o}\zeta^2\left[1 + \zeta Z(\zeta)\right] \tag{A17}$$

$$J_{2l}(\zeta) = 2\left(\frac{v_T}{c}\right)^4 \sin\theta \left(\frac{l\pi}{k_oL}\right)\frac{c}{k_o}\zeta^3\left[Z(\zeta) - 2\zeta^2 Z(\zeta) - 2\zeta\right] \tag{A18}$$

$$J_{3l}(\zeta) = -2\sin\theta\left(\frac{v_T}{c}\right)^2 \frac{1}{k_o}\zeta^2\left[1 + \zeta Z(\zeta)\right] \tag{A19}$$

The identities (A4) and (A5) in connection with equations (A17) to (A19) may be used to express $J_{4l}(\zeta)$ and $J_{5l}(\zeta)$ in terms of the dispersion function. For $l = 0$, the dispersion integrals become

$$J_0 = \frac{J_{50}}{v_T^2} = -\frac{Z(\zeta_o)\omega}{\zeta_o} \tag{A20}$$

$$J_{10} = J_{20} = 0 \tag{A21}$$

$$J_{30} = -\frac{\left[1 + \zeta_o Z(\zeta_o)\right]}{k_o \sin\theta} \tag{A22}$$

$$J_{40} = -\frac{2v_T\zeta_o}{\sqrt{2}\,k_o\sin\theta}\left[\zeta_o Z(\zeta_o) + 1\right] \tag{A23}$$

where

$$\zeta_o = \frac{\omega}{k_o\sqrt{2}\,v_T\sin\theta} \tag{A24}$$

Finally, if H_{ly} had been eliminated in equation (43), the Fourier components of the current density could have been expressed in the following form:

$$j_{lx} = \frac{i\omega_p{}^2\epsilon_o}{v_T{}^2}\left(J_{l5}E_{lx} + iJ_{2l}E_{lz}\right) \tag{A25}$$

$$j_{lz} = \frac{\omega_p{}^2\epsilon_o}{v_T{}^2}\left(iJ_{4l}E_{lz} + J_{2l}E_{lx}\right) \tag{A26}$$

If the modes of the conductivity tensor $\overleftrightarrow{\sigma}$ values are defined through Ohm's law,

$$\left.\begin{aligned} j_{lx} &= \sigma_{xxl}E_{lx} + \sigma_{xzl}E_{lz} \\ j_{lz} &= \sigma_{zxl}E_{lx} + \sigma_{zzl}E_{lz} \end{aligned}\right\} \tag{A27}$$

These expressions agree with those of Hinton, who solved this problem by integrating over particle trajectories.

For a finite collision frequency ratio ν/ω, the pertinent dispersion integrals are given by equations (56), (57), (59), and (60), which are also expressible in terms of the dispersion function as

$$J_{1l} = -\left(1 + i\frac{\nu}{\omega}\right)\frac{2l\pi}{k_oL}\left(\frac{v_T}{c}\right)^2 \frac{1}{k_o}(\zeta')^2\left[1 + \zeta'Z(\zeta)\right] \tag{A28}$$

$$J_{2l} = 2\left(\frac{v_T}{c}\right)^4 \sin\theta\left(\frac{l\pi}{k_oL}\right)\frac{c}{k_o}(\zeta')^3\left[Z(\zeta') - 2(\zeta')^2Z(\zeta') - 2\zeta'\right] \tag{A29}$$

$$J_{3l} = -2\left(1 + i\frac{\nu}{\omega}\right)\sin\theta\left(\frac{v_T}{c}\right)^2 \frac{1}{k_o}(\zeta')^2\left[1 + \zeta'Z(\zeta')\right] \tag{A30}$$

$$J_{4l} = \frac{c}{\sin\theta}\left(J_{3l} - \frac{l\pi}{k_oL}\frac{1}{c}J_{2l}\right) \tag{A31}$$

where, for a finite collision frequency,

$$\zeta' = \zeta\left(1 + i\frac{\nu}{\omega}\right) \tag{A32}$$

APPENDIX B

INTERACTION OF A PLANE WAVE WITH A PLASMA HALF SPACE

For the case where the slab thickness L approaches infinity, the problem can be solved in the original way that Landau and others have done. The Vlasov equation is written in the form:

$$-i\omega f_1 + v_x \frac{\partial f_1}{\partial x} + ik_o f_1 v_z \sin\theta - \frac{e}{m} E_z \frac{\partial f_o}{\partial v_z} - \frac{e}{m} E_x \frac{\partial f_o}{\partial v_x} = 0 \tag{B1}$$

The Vlasov equation is again Fourier analyzed in configuration space, except that the half-space requires a superposition of continuous modes rather than discrete modes. In other words, f_1 is expanded as a Fourier integral instead of a Fourier series:

$$f_1\left(\vec{v},x\right) = \int_{-\infty}^{\infty} \bar{f}_1\left(\vec{v},k_x\right) e^{ik_x x}\, dk_x \tag{B2}$$

A straightforward Fourier analysis of equation (B1), in connection with the convolution theorem gives:

$$\left.\begin{aligned} f_1 &= \frac{-e}{mv_x}\int_x^{\infty} dx'\, e^{ib(x-x')}\left[\frac{\partial f_o}{\partial v_z} E_z(x') + \frac{\partial f_o}{\partial v_x} E_x(x')\right] \qquad \left(v_x < 0\right) \\ f_1 &= \frac{e}{mv_x}\int_{-\infty}^{x} dx'\, e^{ib(x-x')}\left[\frac{\partial f_o}{\partial v_z} E_z(x') + \frac{\partial f_o}{\partial v_x} E_x(x')\right] \qquad \left(v_x > 0\right) \end{aligned}\right\} \tag{B3}$$

with

$$b = \frac{\omega - k_o v_z \sin\theta}{v_x} \tag{B4}$$

If the field components of equation (B3) are expressed in terms of their Fourier transforms,

$$\bar{f}_1 = \frac{n_o e}{imv_T{}^2} \frac{F_o\left(v_z\bar{E}_z + v_x\bar{E}_x\right)}{\omega - k_o v_z \sin\theta - k_x v_x} \tag{B5}$$

for all v_x where a bar denotes the Fourier transform. From the definition of the current components,

$$\bar{j}_x = \frac{i\epsilon_o\omega_p{}^2}{v_T{}^2}\left[\bar{E}_z J_2\left(k_x\right) + \bar{E}_x J_5\left(k_x\right)\right] \tag{B6}$$

$$\bar{j}_z = \frac{i\epsilon_o\omega_p{}^2}{v_T{}^2}\left[\bar{E}_z J_4\left(k_x\right) + \bar{E}_x J_2\left(k_x\right)\right] \tag{B7}$$

where the J terms in equations (B6) and (B7) are identical to the dispersion integrals given in appendix A, $l\pi/L$ being replaced by the continuous wave number k_x. The currents explicitly appear in the wave equation as follows:

$$\frac{d^2H_y}{dx^2} + \left(k_o{}^2 - k_o{}^2 \sin^2\theta\right)H_y = -ik_z j_x + \frac{dj_z}{dx} \tag{B8}$$

The form of equation (B3) suggests the symmetry properties $H_y(x) = -H_y(-x)$, $j_x(x) = -j_x(-x)$, and $j_z(x) = j_z(-x)$. Applying these properties to the Fourier transform (eq. (B8)) gives

$$\left(k_o{}^2 - k_o{}^2 \sin^2\theta - k_x{}^2\right)\bar{H}_y - ik_x \frac{H_y(0)}{\pi}$$

$$= \frac{\epsilon_o\omega_p{}^2}{v_T{}^2}\,\bar{E}_x\left[-k_x J_2\left(k_x\right) + k_o \sin\theta\, J_5\left(k_x\right)\right] + \frac{\epsilon_o\omega_p{}^2}{v_T{}^2}\,\bar{E}_z\left[-k_x J_4\left(k_x\right) + k_o \sin\theta\, J_2\left(k_x\right)\right] \tag{B9}$$

where a bar over a field component denotes a Fourier transform. The Fourier transforms of the Maxwell curl equations are:

$$ik_x\bar{E}_z = ik_o \sin\theta\, \bar{E}_x - i\omega\mu_o\bar{H}_y \tag{B10}$$

$$\overline{E}_x = \frac{k_o \sin\theta}{\omega\epsilon_o}\overline{H}_y + \frac{1}{i\omega\epsilon_o}\left\{\frac{i\omega_p{}^2\epsilon_o}{v_T{}^2}\left[\overline{E}_z J_2(k_x) + \overline{E}_x J_5(k_x)\right]\right\} \tag{B11}$$

Equations (B9), (B10), and (B11) are all that are needed to solve for all the field components. After several ponderous algebraic manipulations, the following results are obtained:

$$\overline{H}_y = \frac{ik_x H_y(0)}{\pi\left\{k_o{}^2 - k_o{}^2\sin^2\theta - k_x{}^2 - \frac{\omega_p{}^2 k_o{}^2}{\omega v_T{}^2}\left[J_4(k_x) - \frac{k_o \sin\theta}{k_x}J_2(k_x)\right]\right\}} \tag{B12}$$

$$\overline{E}_x = \frac{ik_x H_y(0)\left[k_o \sin\theta - \frac{\omega_p{}^2}{v_T{}^2}\frac{k_o}{c}\frac{J_2(k_x)}{k_x}\right]}{\omega\epsilon_o\pi\left[1 - \frac{\omega_p{}^2}{v_T{}^2}\frac{J_1(k_x)}{k_x}\right]\left\{k_o{}^2 - k_o{}^2\sin^2\theta - k_x{}^2 - \frac{\omega_p{}^2 k_o{}^2}{\omega v_T{}^2}\left[J_4(k_x) - \frac{k_o \sin\theta}{k_x}J_2(k_x)\right]\right\}} \tag{B13}$$

$$\overline{E}_z = \frac{-ik_x{}^2\left[1 - \sin^2\theta - \frac{\omega_p{}^2}{\omega v_T{}^2}J_5(k_x)\right]H_y(0)\,\omega\mu_o}{\pi\left[1 - \frac{\omega_p{}^2}{v_T{}^2}\frac{J_1(k_x)}{k_x}\right]\left\{k_o{}^2 - k_o{}^2\sin^2\theta - k_x{}^2 - \frac{\omega_p{}^2 k_o{}^2}{\omega v_T{}^2}\left[J_4(k_x) - \frac{k_o \sin\theta}{k_x}J_2(k_x)\right]\right\}} \tag{B14}$$

The field expressions for the plasma slab (eqs. (54) to (56)) are of the following form:

$$H_y = \sum_{l=1}^{\infty} H_{ly} \sin\frac{l\pi x}{L} \tag{B15}$$

$$E_x = \sum_{l=1}^{\infty} E_{lx} \sin\frac{l\pi x}{L} \tag{B16}$$

$$E_z = \sum_{l=0}^{\infty} \frac{E_{lz}}{1 + \delta_o{}^l} \cos\frac{l\pi x}{L} \tag{B17}$$

But, it also follows from equations (54) to (56) that $H_{ly} = -H_{-ly}$, $E_{lx} = -E_{-lx}$ and $E_{lz}' = E_{-lz}$ so that the summation index l can be extended from $-\infty$ to $+\infty$ to yield

$$H_y = \frac{1}{2i} \sum_{-\infty}^{\infty} H_{ly}\, e^{\frac{il\pi x}{L}}$$

and similar expressions for E_x and E_z. In the limit as L approaches ∞, the summations convert to integrals so that $l\pi/L$ approaches k_x and dk_x approaches π/L. In this limit,

$$H_y = \frac{1}{2i} \sum_{-\infty}^{\infty} H_{ly}\, e^{\frac{il\pi x}{L}} \longrightarrow \frac{L}{2\pi i} \int_{-\infty}^{\infty} H_y(k_x)\, e^{ik_x x}\, dk_x \tag{B18}$$

In which case, the sum of the terms defining H_y in equation (54) becomes

$$H_y \longrightarrow - \int_{-\infty}^{\infty} \frac{dk_x\, e^{ik_x x} k_x H_y(0)}{\pi i \left\{k_o^2 - k_o^2 \sin^2\theta - k_x^2 - \frac{k_o^2 \omega \omega_p^2}{\omega^2 v_T^2}\left[J_4(k_x) - \frac{k_o \sin\theta}{k_x} J_2(k_x)\right]\right\}} \tag{B19}$$

where $H_y(L)$ approaches 0 as L approaches ∞. Equation (B19) is the Fourier transform of equation (B12). Since equations (B13) and (B14) follow in a similar manner, it has been proven that the results for the slab reduce to the half-space results as L approaches ∞, which was to be shown.

APPENDIX C

SOME COMMENTS ON THE COMPUTATIONAL PROCEDURE

The programs for all numerical computations were written in Fortran IV for use on the Control Data 6600 computer facility at the Langley Research Center. The programs applicable to the cold plasma and hydrodynamic models involve only the elementary transcendental functions of complex arguments and therefore do not require any detailed explanation. The programs relating to the Vlasov equation should, however, be discussed in a little more detail. The reflection, transmission, and absorption coefficients were determined by evaluating the series defining G_1 and G_2 (eqs. (58) and (59)), which are repeated here again for reference:

$$G_1 = \frac{1}{(k_oL)^2} \sum_{l=0}^{\infty} \frac{2}{1+\delta_o^l} \frac{1 - \sin^2\theta - \frac{\omega\omega_p^2}{v_T^2\omega^2} J_{5l}}{\left(1 - \frac{\omega_p^2}{v_T^2} \frac{L}{l\pi} J_{1l}\right)\left[1 - \sin^2\theta - \left(\frac{l\pi}{k_oL}\right)^2 - \frac{\omega\omega_p^2}{v_T^2\omega^2}\left(J_{4l} - \frac{k_oL \sin\theta}{l\pi} J_{2l}\right)\right]}$$

$$G_2 = \frac{1}{(k_oL)^2} \sum_{l=0}^{\infty} \frac{2}{1+\delta_o^l} \frac{\left(1 - \sin^2\theta - \frac{\omega\omega_p^2}{v_T^2\omega^2} J_{5l}\right)(-1)^l}{\left(1 - \frac{\omega_p^2}{v_T^2} \frac{L}{l\pi} J_{1l}\right)\left[1 - \sin^2\theta - \left(\frac{l\pi}{k_oL}\right)^2 - \frac{\omega\omega_p^2}{v_T^2\omega^2}\left(J_{4l} - \frac{k_oL \sin\theta}{l\pi} J_{2l}\right)\right]}$$

Equations (58) and (59) are then used in a trivial manner in connection with equations (15), (16), (17), and (18) to solve for the reflection, transmission, and absorption coefficients. The dispersion integrals (the J terms) in equations (58) and (59)) were then expressed in terms of the dispersion function $Z(\zeta)$, which for $\zeta < 4$ was computed from the differential equation:

$$Z'(\zeta) = -2\left[1 + \zeta Z(\zeta)\right] \tag{C1}$$

under the initial condition that $Z(0) = i\sqrt{\pi}$. For $\zeta > 4$, the asymptotic series was used (ref. 23). Calculations were performed only for real values of ζ

$$\zeta_R = \frac{1}{\sqrt{2}\frac{v_T}{c}\sqrt{\frac{l^2\pi^2}{k_o^2L^2} + \sin^2\theta}}$$

APPENDIX C

For small ν/ω, the dispersion function was evaluated by using a Taylor series expansion:

$$Z(\zeta) \approx Z(\zeta)\Big|_{\frac{\nu}{\omega}=0} + \frac{dZ(\zeta)}{d\left(\frac{\nu}{\omega}\right)}\Bigg|_{\frac{\nu}{\omega}=0} \frac{\nu}{\omega} \tag{C2}$$

where, from equation (A32) $\zeta = \zeta^R\left(1 + i\frac{\nu}{\omega}\right)$. Equation (C2) therefore reduces to

$$Z(\zeta) \approx Z(\zeta) - 2i\left[1 + \zeta Z(\zeta)\right] \zeta \frac{\nu}{\omega} \tag{C3}$$

for $\frac{\nu}{\omega} << 1$.

For plasma slabs which are thin compared with a wavelength, the numerical results indicated that G_1 and G_2 converged so rapidly that only the $l = 0$ term and the most nearly resonant term contribute to the series for G_1 and G_2. The resonant term is that term in odd l for which $1 - \frac{\omega_p^2}{v_T^2} \frac{L}{l\pi} J_{1l}$ in the denominator of equations (68) and (69) is a minimum. The dominant $l = 0$ term is simply

$$G_1(l=0) = G_2(l=0) = \frac{1}{k_o^2 L^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right)} \tag{C4}$$

REFERENCES

1. Swift, Calvin T.; and Evans, John S: Generalized Treatment of Plane Electromagnetic Waves Passing Through an Isotropic Inhomogeneous Plasma Slab at Arbitrary Angles of Incidence. NASA TR R-172, 1963.

2. Swift, Calvin T.: Radiation Patterns of a Slotted-Cylinder Antenna in the Presence of an Inhomogeneous Lossy Plasma. IEEE Trans. Antennas Propagation, vol. AP-12, no. 6, Nov. 1964, pp. 728-738.

3. Wait, James R.: Radiation From Sources Immersed in Compressible Plasma Media. Can. J. Phys., vol. 42, no. 9, Sept. 1964, pp. 1760-1780.

4. Tonks, Lewi: Plasma-Electron Resonance, Plasma Resonance and Plasma Shape. Phys. Rev., Second ser., vol. 38, no. 6, Sept. 15, 1931, pp. 1219-1233.

5. Romell, Dag: Radio Reflexions From a Column of Ionized Gas. Nature, vol. 167, no. 4241, Feb. 10, 1951, p. 243.

6. Dattner, Adam: The Plasma Resonator. Ericsson Technics, vol. 13, no. 2, 1957, pp. 309-350.

7. Parker, J. V.; Nickel, J. C.; and Gould, R. W.: Resonance Oscillations in a Hot Nonuniform Plasma. Phys. Fluids, vol. 7, no. 9, Sept. 1964, pp. 1489-1500.

8. Landau, L.: On the Vibrations of the Electronic Plasma. J. Phys., vol. X, no. 1, 1946, pp. 25-34.

9. Bailey, Marion C.; and Swift, Calvin T.: Input Admittance of a Circular Waveguide Aperture Covered by a Dielectric Slab. IEEE Trans. Antennas Propagation, vol. AP-16, no. 4, July 1968, pp. 386-391.

10. Hinton, F. L.: Collisionless Absorption and Emission of Electromagnetic Waves by a Bounded Plasma. AFOSR 67-1122, U.S. Air Force, May 1967. (Available from DDC as AD 654 314.)

11. Bowman, J. J.; and Weston, V. H.: Oblique Incidence of an Electromagnetic Wave on a Plasma Layer. Phys. Fluids, vol. 11, no. 3, Mar. 1968, pp. 601-606.

12. Kondratenko, A. N.; and Miroshnichenko, V. I.: Kinetic Theory of the Propagation of Electromagnetic Waves Through a Plasma Layer. II. Sov. Phys.-Tech. Phys., vol. 11, no. 1, July 1966, pp. 16-21.

13. Case, K. M.; Plasma Oscillations. Ann. Phys. (N.Y.), vol. 7, no. 3, July 1959, pp. 349-364.

14. Shure, Fred Charles: Boundary Value Problems in Plasma Oscillations. Ph. D. Diss., Univ. of Michigan, [1962].

15. Felderhof, B. U.: Theory of Transverse Waves in Vlasov-Plasmas – III. Reflection of Electromagnetic Waves by a Plasma Half-Space. Physica, vol. 29, 1963, pp. 662-674.

16. Van Kampen, N. G.: On the Theory of Stationary Waves in Plasmas. Physica, vol. 21, no. 12, Dec. 1955, pp. 949-963.

17. Melnyk, Andrew Rostyslaw: The Interaction of Electromagnetic Radiation With a Bounded Plasma. Ph. D. Thesis, Michigan State Univ., 1967.

18. Hall, Richard B.: Parallel-Plate Plasma Problem. Amer. J. Phys., vol. 31, no. 9, Sept. 1963, pp. 696-706.

19. Bhatnagar, P. L.; Gross, E. P.; and Krook, M.: A Model for Collision Processes in Gases. I. Small Amplitude Processes in Charged and Neutral One-Component Systems. Phys. Rev., Second ser., vol. 94, no. 3, May 1, 1954, pp. 511-525.

20. Stratton, Julius Adams: Electromagnetic Theory. McGraw-Hill Book Co., Inc., 1941.

21. Huang, Kerson: Statistical Mechanics. John Wiley & Sons, Inc., c.1963.

22. Diament, Paul: Inverse Velocity Space Spectra and Kinetic Equations. Amer. J. Phys., vol. 35, no. 10, Oct. 1967, pp. 906-912.

23. Fried, Burton D.; and Conte, Samuel D.: The Plasma Dispersion Function. Academic Press, Inc., 1961.

24. Shure, F. C.: Boundary Value Problems in Plasma Oscillations: The Plasma Capacitor. J. Nucl. Energy: Pt. C, vol. 6, no. 1, Jan./Feb. 1964, pp. 1-14.

25. Leavens, William M.: Scattering Resonances of a Cylindrical Plasma. J. Res. Nat. Bur. Stand., D, vol. 69D, no. 10, Oct. 1965, pp. 1321-1333.

26. Staton, Leo Douglas: Electrostatic Oscillations in Inhomogeneous Plasmas. Ph. D. Thesis, Virginia Polytech. Inst., 1968.

27. Ignat, David W.: Resonant Behavior of a Bounded Plasma. Ph. D. Diss., Yale Univ., 1969.

28. Jackson, E. A.; and Raether, M.: Oscillations in a One-Dimensional, Inhomogeneous Plasma. Rep. R-239 (Contract DA-28-043-AMC-00073(E)), Univ. of Illinois, Jan. 1965.

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